CHALLENGING FACT FAMILY REASONING WITH INSTRUCTION IN NEGATIVE NUMBERS

Laura Bofferding
lboff@stanford.edu
Stanford University

Through "fact family" instruction, students learn the commutative property of addition and inversion principle. However, problems like 3-9, challenge students' understanding of these principles due to their novel arrangement. Sixty-one first graders were randomly assigned to one of three instructional groups. Pre- and post-test interviews indicate that students who practiced operating through zero had greater gains in principle understanding.

Introduction

One of the fundamental goals of elementary mathematics instruction is to help students develop number sense; for children in grades K-2, in particular, number sense involves understanding the composition and decomposition of numbers as well as the relationships among numbers in addition and subtraction problems (National Council of Teachers of Mathematics, 2000). An important part of this process includes learning three principles: the commutative property of addition (e.g., 4+5 = 5+4), the inversion principle (e.g., 4+5=9, so 9-5=4), and the subtraction complement principle (e.g., 9-4=5 so 9-5=4). Knowledge of additive commutativity tends to appear before the other two principles, with inversion presenting more difficulty for young children (Canobi, 2005).

Instead of only addressing these part-whole relationships individually, US textbooks frequently incorporate a series of lessons on "fact families" which aim to help students see these connections all at once (Lovin, 2006). In the California version of the first grade *enVision* mathematics curriculum, there is one lesson dedicated to the commutative property of addition and seven lessons on relating addition to subtraction and using "fact families" (Pearson Education, Inc., 2009). Although the use of "fact families" or "turn-around facts" as an instructional focus seems widespread, there is little research on their effectiveness. Because studies show students understand that 4+5=5+4 and teachers see students correctly filling in the "fact families", we might believe that students *do* understand all of these principles; however, other studies show that students solve problems like 62–48 by subtracting the smaller number from the larger number in each column, regardless of their placement (Fuson, 2003). These results suggest that students may not have full understanding of these principles. This paper takes a small step at exploring how early instruction in negative numbers might facilitate first grade students' judgment of when and how to use the commutative property and inversion principle.

Theoretical Framework

As children learn addition and subtraction and develop more efficient strategies for solving arithmetic problems, they move through three "conception of quantities" levels (and sometimes a transition level) (Fuson, 1992; Murata, 2004). At Level 1, students count all quantities; for 3+4 a child would count out three objects, count out four objects, and count the combined collection to find the total. Students at Level 2 shorten this process by counting on (or counting back or up for subtraction); for 3+4, this student would say "three", knowing that it is not necessary to count out the three, and then count on "four, five, six, seven." Finally, at Level 3, students use composition and decomposition methods, especially involving groups of ten, to solve problems. Learning and using these part-whole relationships, such as those emphasized with fact families, is one of the most important goals in elementary arithmetic (National Research Council, 2001).

Aside from guiding students towards part-whole strategies, effective mathematics instruction also needs to help students develop mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (National Research Council, 2001). According to the National Research Council, "Examining the relationships between addition and subtraction and seeing subtraction as involving a known and an unknown addend are examples of adaptive reasoning" (p. 191). Along with fostering adaptive reasoning, presenting all forms of a fact family together (e.g. 4+5=9, 5+4=9, 9-5=4, and 9-4=5) might also help students deepen their *conceptual understanding* of the commutative property, subtraction complement principle, and inverse principle as students make connections among these relationships. However, by spending limited to no time learning the principles separately, it is also possible that rather than understanding these complex relationships, students learn that order does not matter for addition and likewise assume that the numbers in subtraction problems can be moved around in multiple ways.

A more *procedural understanding* of these principles could lead students to incorrectly think they can solve 2–7 as they would 7–2 by misapplying the commutative property or by thinking, "What plus 2 equals 7?" due to a misapplication of the inversion principle. Alternatively, students may use these misapplied methods because they do not know about negative numbers but feel the need to provide an answer anyway. Students who know about negative numbers and their relationship to positive numbers may be willing to think about obtaining answers other than positive numbers. Furthermore, learning that the commutative property does not hold for subtraction and having experiences correctly solving subtraction problems with smaller minuends might help these students better understand both the commutative property and the inversion principle.

Research Questions

The following questions guided this analysis:

1) How do students reason about the commutative and inversion properties before and after instruction in negative numbers?

2) How is instruction on the order and value of negatives and/or addition and subtraction with negatives related to students' understanding of commutativity and inversion?

Methods

Subjects and Site

This data comes from a study conducted at an elementary school in northern California, in which 47% of its students are English language learners (California Department of Education, 2010). Out of a possible 79 first graders at the school, 61 first graders (30 male, 31 female) agreed to participate and complete the interviews in English. The study took place in the spring, so the first graders had already learned about addition and subtraction.

Materials and Data Collection

The study employed a pre-test, post-test design with an instructional intervention. Both pre- and post-tests were conducted as individual interviews and involved similar questions; some post-test problems were identical to those on the pre-test, while others just had different numbers. During the interviews, students were asked to explain how they solved the problems.

While the questions covered a wide-range of integer concepts, including addition and subtraction problems, this paper focuses on a small-subset of question categories: counting backward, integer value comparisons, commutative property, and subtraction problems. Table 1 lists the questions from both the pre- and post-tests.

Table 1. Items students completed during the pre-test and post-test interviews.

Question Category	Pre-Test	Post-Test
Counting backward	"Start at five and count backwards as far as you can. Is there anything before <last number child says>?"	"Start at five and count backwards as far as you can. Is there anything before <last number child says>?"
Integer value comparison *"What are these two numbers? Circle the one that is greater."*	8 vs. 6*	6 vs. 4*
	3 vs. -9	5 vs. -7
	-2 vs. -7	-3 vs. -1
	-5 vs. 3	-8 vs. 4
	-8 vs. -2	-6 vs. -2
"Two children are playing a game and trying to get the highest score. Circle who's winning."	4 vs. -7	5 vs. -9
	-7 vs. -3	-8 vs. -6
Commutative property *"Just look at these two problems. Do you think they will give you the same or different answers?"*	4 + 5 vs. 5 + 4	2 + 5 vs. 5 + 2
	3 – 1 vs. 1 – 3	4 – 1 vs. 1 – 4
	6 – 4 vs. 7 – 4 (distracter)	7 – 3 vs. 7 – 4 (distracter)
	5 – 8 vs. 8 – 5	2 – 9 vs. 9 – 2
	3 + 2 vs. 3 + 3 (distracter)	6 + 3 vs. 2 + 6 (distracter)
	9 – 6 vs. 6 – 9	0 – 8 vs. 8 – 0
Subtraction *"Solve this problem."*	1 – 4 =	1 – 4 =
	3 – 9 =	3 – 9 =
	6 – 8 =	6 – 8 =

*All students solved these problems correctly, so they were excluded from further analysis.

After the pre-test, students (regardless of classroom) were randomly assigned to one of three instructional groups, so that each group had an even mix of students (in terms of initial understanding of negative numbers, teacher ratings of their math performance, and gender). Each group participated in 8, 45-minute lessons. During their group's instructional time, students met in a separate room, and the author provided instruction. Group 1 (N=20) received instruction on the order and value of negative numbers along with how to add and subtract with them; this included learning that subtraction is not commutative. Group 2 (N=21) only received the "adding and subtracting" lessons from Group 1's instruction, without learning about the order and value of negatives, while Group 3 (N=20) only received the "order and value" lessons from Group 1's instruction. Both Group 2 and Group 3 had additional practice games similar to their original lessons so that they received the same amount of lesson time as Group 1. Table 2 provides an outline of the lessons for the three groups.

Table 2. Lesson topics for the three instructional groups.

	Day 1	Day 2	Day 3
Group 1 (N=20) Integer Value & Order, Add, Subtract	Discuss and explore symbols	Explore minus sign versus negative sign	Vertical number line with integers; Game: Which is greater?
Group 2 (N=21) Add, Subtract	Explore lack of commutativity for subtraction compared to addition, no specific mention of negatives		
Group 3 (N=20) Integer Value & Order	Discuss and explore symbols	Explore minus sign versus negative sign	Match negative numbers vs. positive numbers

	Day 4	Day 5	Day 6	Day 7	Day 8
Group 1 (cont.)	Explore commutative property and lack of commutativity for subtraction compared to addition		More (move right) vs. Less (move left)	More positive = larger, more negative = smaller	Less positive = smaller, less negative = larger
Group 2 (cont.)	More (go right on NL) versus Less (go left on NL)	More positive= larger, more negative= smaller	Less positive= smaller, less negative= larger	More positive, less positive, more negative, less negative	
Group 3 (cont.)	Negatives on a vertical number line; Game: Which is greater?		War: Which integer is Greater?	Game: Get three or four consecutive integers in a row	

Analysis

Students' solutions were coded for number correct in each category, and their verbal reports were transcribed for each. When counting backward, students only had to count to "negative one" in order for their count to be considered correct. Using the terminology "minus one" or "penalty one" also counted. Regarding the commutative property questions, all distracter questions were removed from this analysis (only three students gave an incorrect answer for one or both of these). Students' explanations were coded based on their reasoning for why they thought the answers would be the same or different (e.g., Commute = commenting on the numbers being switched around, Inverse = justifying a subtraction problem based on an addition problem). Finally, for the subtraction problems, answering "0" or a negative number was counted as correct since these answers meant students were not reversing the order of the numerals, which was all the study explored. The explanations of students' solutions were coded as to whether they reversed the numbers or used inversion reasoning in solving the problems.

Results

Table 3 lists each group's percentage correct scores on the pre- and post-tests for each question category, along with their overall percentage gains. On the counting backward and integer value comparison tasks, both groups who had instruction on integer order and values (Groups 1 and 3) made pre- to post-test gains that are almost double the gain of Group 2, who did not receive this instruction. Unsurprisingly, on the pre-test, all groups demonstrated understanding of the commutative property of addition, judging that 4+5 would give them the same answer as 5+4; because of their high initial performance on this item, there were small to no gains on this item across groups.

When asked if subtraction problems and their reversals would give them the same answer, Group 3—who learned about the order and value of negatives—made a large gain in understanding that the answers would be different. However, when they were asked to solve the problems, they made no gain in providing negative or zero answers. On the flip side, Group 1—who received instruction in all of the topics—made no gain in identifying that the subtraction problems with smaller minuends would have different answers but made a modest gain in actually providing negative or zero answers when they had to solve them. Group 2—who practiced adding and subtracting on both sides of zero—improved on both of these question types.

Table 3. *Percentages correct (and gains) for each group on pre- and post-tests by question type.*

Counts Backward into the Negatives	**PreTest (%)**	**PostTest (%)**	**% Gain**
Group 1 (N=20)	15%	60%	**45%**
Group 2 (N=21)	14%	38%	24%
Group 3 (N=20)	20%	65%	**45%**
Correctly Compares Integer Values	**PreTest (%)**	**PostTest (%)**	**% Gain**
Group 1 (N=20)	18%	73%	**55%**
Group 2 (N=21)	17%	32%	15%
Group 3 (N=20)	20%	90%	**70%**
Commutative Property of Addition	**PreTest (%)**	**PostTest (%)**	**% Gain**
Group 1 (N=20)	85%	95%	10%
Group 2 (N=21)	90%	95%	5%
Group 3 (N=20)	90%	90%	0%
No Commutative Property of Subtraction	**PreTest (%)**	**PostTest (%)**	**% Gain**
Group 1 (N=20)	37%	32%	-3%
Group 2 (N=21)	11%	37%	**25%**
Group 3 (N=20)	15%	48%	**33%**
Solving S – L, where L>S, S, L > 0	**PreTest (%)**	**PostTest (%)**	**% Gain**
Group 1 (N=20)	33%	45%	12%
Group 2 (N=21)	38%	63%	**25%**
Group 3 (N=20)	47%	45%	-2%

Students' Responses

While Group 2 was the only group to show growth for both the commutative questions and the subtraction problems, there were similar trends and student variation in each of the groups. Over all groups, students reversed numbers to solve 3-9, 6-8, and 1-4. An additional nine students specifically misapplied inversion reasoning to justify their positive answers; this is likely an underestimate since several students provided positive answers without justifying why they did so. Each group had at least one student who originally claimed problems such as 3-9 and 9-3 would have different answers but then on the post-test stated they would have the same answers. Furthermore, each group had students who demonstrated no gains from pre- to post-test (although some of these students changed their reasoning about their answers). Finally, all groups contained students who improved in realizing that the subtraction problems would give them different answers, and more specifically, that subtraction problems with smaller minuends would not have positive answers. See Table 4 for examples of how students in these subsets reasoned from pre- to post-test. The examples are from students in Group 1, but their reasoning is reflective of the other groups' reasoning.

Table 4. Examples of students from Group 1 whose performance decreased, stayed the same with different reasoning, and improved from pre- to post-test.

ID	Test	Answer	Explanation	Code
209	Pre-Test	3 – 1 ≠ 1 – 3	Three and taking one is (writes 2). There's one and you took away three and there's no more, and it's zero (writes 0).	Zero / Positive
	Post-Test	4 – 1 = 1 – 4	Four. Four. One. One.	Same Numbers
213	Pre-Test	3 – 1 ≠ 1 – 3	When you minus, it can't equal the same number.	Not Equal
	Post-Test	4 – 1 ≠ 1 – 4	This one [1 – 4] would equal a negative and this one [4 – 1] would equal a positive number.	Negative/ Positive
101	Pre-Test	3 – 9 = 6	Six plus three is nine.	Inversion
	Post-Test	3 – 9 = -6	(counted back on fingers) Two, one, zero, negative one, negative two, negative three, negative four, negative five, negative six.	Count Through Zero

Discussion and Conclusions

While all of the groups improved in a couple areas, some of the findings are clearer than others. The differential instruction influenced students' improvement on certain questions due to their emerging understanding. The two instructional groups who learned about the order and value of negative numbers (Groups 1 and 3) had greater gains on the counting backward and integer comparison tasks. This is unsurprising since students in these groups practiced counting backward through zero and played games which focused on the value of integers. Group 3, on the other hand, only saw negatives written on a number line in one lesson, and the numbers were not named or pointed out to them.

As found in previous studies, these first graders also showed consistent knowledge of the commutative property of addition. The patterns for the subtraction problems with smaller minuends are less clear. Group 2 showed gains in identifying that subtraction problems, when reversed, will result in different answers; they also made gains applying this knowledge in order to solve the subtraction problems. They transitioned from giving positive answers to answering mostly negative numbers or zero. During instruction, this group spent several lessons playing games where they moved back and forth across zero as they acted out addition and subtraction problems. These motions may have facilitated their developing conceptual understanding of why the answers would be different and helped them avoid the perceptual inclination to think that commuting in subtraction is okay because the problems contain the same numbers.

Group 1, however, also received this movement instruction, but their gain was much smaller than Group 2's. A possible reason for this is that Group 1 spent less time on the addition and subtraction activities because they also had lessons on integer values. It is reasonable that the length of time students practice moving beyond zero is related to their ability to reason about why subtraction problems with smaller minuends would have different answers than their reversals. The results of this study only hint at a possible connection, but further study is needed.

Although Group 1 did slightly improve in concluding that the subtraction problems with smaller minuends would not be positive, they did not improve in judging that subtraction is not commutative. How can we account for these contradictory results? On the one hand, more students in this group started with knowledge that subtraction is not commutative compared to the other groups, so they had less room for growth. On the other hand, four of the students had lower performance on the post-test, so it is possible that the combination of both types of instruction was too much to keep straight in such a short period of time. Again, investigating instruction over a longer time period or with slightly older students who may be better able to integrate the many aspects of the lessons may provide insight into this issue.

Finally, while Group 3 showed gains in identifying that subtraction is not commutative, they did not make any gains in applying this knowledge. Over half of the students could count into the negatives, yet they did not use this strategy to solve the subtraction problems. Furthermore, two students stated that subtracting a larger number from a smaller one would be a negative number, yet they still wrote positive answers to the subtraction problems. One possible explanation for this result is that students are used to inverting problems from working with fact families, and they continue to invert even in contexts where this application is incorrect—even if they know it!—since they did not have the experience (as Group 2 did) breaking out of this habit.

A second possible explanation is that students may be able to operate at Level 3 of the conception of quantities either on a procedural/perceptual level *or* on a conceptual level. One hypothesis was that students in Group 3 should have been more willing to get negative answers because they had instruction in their existence. However, they also may have had a surface understanding of the commutative and inverse properties—believing that one can change the order of the numbers in any way. Since they did not have instruction in solving negative problems, students in this group may have felt it easier to use one of the properties they "knew"—such as "switching" the numbers—rather than attempting to solve the problems in a new way.

Clearly, the results of this study raise more questions than provide answers, and we need to investigate children's learning of the commutative and inverse properties further. However, one noteworthy finding is that if children have knowledge of the commutative property of addition this does not mean that they understand its limits. A stronger argument could be made for the inversion principle. By using 2+3=5 to solve 3-5, children demonstrate that the numbers, rather than their order and relationship to each other are more salient features to them. These results suggest that students will develop deeper conceptual understanding as well as the flexibility to know when they can apply learned rules and reasoning by having experiences with those situations which require them to expand or challenge their original understanding. If we can help students develop deeper schemas for concepts and procedures earlier, they will have a stronger mathematical base on which to build future concepts.

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