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Negative Integer Understanding: Characterizing First Graders' Mental Models

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This paper presents results of a research study with a pretest, instruction, posttest design that aimed to identify students' mental models of integers and investigate how these models change based on instruction on the unary and/or binary meanings of the minus sign. Sixty-one first graders' responses to interview questions about negative integer values and order and directed magnitudes were examined to characterize the students' mental models. The models reveal that initially, students over-relied on various combinations of whole number principles as they tried to understand negative integers. However, a significant number of students developed formal mental models based on their instruction.

Key words: Central conceptual structure; Conceptual change; Elementary; Mental models; Negative integers

Researcher: *Why do you think the negatives are so tricky to think about?*

Ashvin (4th grader): *Because you don't learn them until like fourth grade, and then you're so used to regular numbers.* (Interview, July 7, 2009)

As captured by Ashvin, many students struggle to make sense of negative integer concepts because they seemingly conflict with their established understanding of non-negative numbers. Students start learning whole number concepts in kindergarten (or earlier), fractions in second or third grade, and decimals in fourth grade but, typically, do not learn negative number concepts until sixth and seventh grade in the United States (National Governor's Association for Best Practices & Council of Chief State School Officers, 2010) and other countries, including Singapore (Ministry of Education, 2012; see also Ginsburg, Leinwand, & Decker, 2009) and the Netherlands (van den Heuvel-Panhuizen & Wijers, 2005). In order to avoid referencing negative numbers during the gap between these grades, many elementary teachers tell students to always subtract with the larger number first, and preservice teachers often think this is a good rule to share (Ball & Wilson, 1990). The *larger minus smaller* rule contributes to students solving problems such as $24 - 19$ as $29 - 14$ because they switch the digits in the ones place (Fuson, 2003). Even without hearing this rule, many students incorrectly solve problems such as $3 - 5$ as $5 - 3$. Moreover, this problem persists even after students learn about negative numbers and can solve other problems with them (Bofferding, 2011b; Murray, 1985). Knowing more about younger students' potential to understand negatives could enable teachers to build on students' understanding rather than contradict it.

Integer Conceptions

One way to interpret negative numbers is as directed magnitudes. Although directed magnitudes build on whole number ideas of magnitude, only with the added element of direction, students may have difficulty coordinating the two aspects. In one study, Thompson and Dreyfus (1988) found that two sixth graders initially thought about the magnitude of a turtle's movement along a number line (e.g., 30 paces) separate from the direction in which it would move (e.g., in the negative direction). The students participated in a constructivist teaching experiment that involved exploring integer problems in a microworld context for eleven 40-minute sessions over 6 weeks. By the third session, they began to discuss the integer movements as directed magnitudes (e.g., negative 30 paces). Whitacre et al. (2012a) argued that reasoning about opposite magnitudes might serve as a foundation for integer reasoning. They asked students what kind of a day a person was having if she had a certain number of happy and sad thoughts. One first grader identified what type of day it was based on whether there were more happy or sad thoughts (e.g., five happy thoughts are more than two sad thoughts, so it would be a happy day overall), while one third grader canceled happy and sad thoughts to specify how many more of one type there were than the other (e.g., two happy and sad thoughts cancel, leaving a three happy-thought day), and one fifth grader used integer notation to show the final result (e.g., 5 + -2 = 3). Whitacre et al. concluded that students across grades K-5 have strong intuitions about opposite magnitudes that reflect varying levels of sophistication.

Integer research findings that focus specifically on number and operations conflict in terms of what students understand about negative numbers across the grades. Based on interviews of students, Peled, Mukhopadhyay, and Resnick (1989) reported that third graders ordered negative numbers correctly, but first graders ordered negative numbers next to their positive counterparts (e.g., 1, -1 , 2, 3, -3) or treated them as zero. When ordering numbers, students who treat negative numbers as zero place them on either side of zero (e.g., 0, -1, -2, -3, 1, 2, 3 or -2, 0, -1, -3, 1, 2, 3) (Schwarz, Kohn, & Resnick, 1993). In contrast, other researchers have reported that some first graders can label numbers before zero, often with their own notation (Aze, 1989; Bishop, Lamb, Phillip, Schapelle, & Whitacre, 2011; Wilcox, 2008). In other interviews, many fourth through seventh graders were able to describe the value of a negative number as a number smaller than zero (Hativa & Cohen, 1995) and could describe how far below zero the number was (Murray, 1985). However, Peled et al. (1989) found that only one first-grade student and just half of the third- and fifth-grade students in their study knew that -4 > -6, and in other studies, some students in fourth (Hativa & Cohen, 1985) and fifth grades (Mukhopadhyay, 1997) had difficulty accepting that values exist below zero.

This variability suggests that experiences play a stronger role in integer understanding than age or developmental levels (at least at the elementary level). In many cases, students rely on their prior knowledge when encountering new problems. For example, students may interpret word problems involving negative amounts by talking about them from a positive perspective. Whitacre et al. (2012b) had students write equations to represent the result of a person borrowing money twice from a friend. Their results indicated that students were likely to interpret integer word problems from the positive perspective, writing equations to show how much money they borrowed (positive) rather than how much money they were in debt (negative). In other cases, exposure to new concepts can help students start to make sense of them. For example, students might hear about negative temperatures and see representations of them, which could prepare them to distinguish between the subtraction and negative signs. As Behrend and Mohs' (2005/2006) work suggests, including negative numbers on a classroom number line can promote students' curiosity about them. From this brief exposure, first graders began incorporating negative numbers into equations and reasoning about adding and subtracting with them without prompting from the teacher.

Integer Notation

One possible reason for the variability in students' integer understanding is their interpretations of integer notation. For example, in multiple studies, when students were asked to label spaces to the left of zero on a board game or number line, some kindergarten and first-grade students appeared to invent notations for negative numbers. Names and notations they used to indicate *negative one* include "zero cousin minus one" or "1-" (Wilcox, 2008, p. 204), "dash one" (Aze 1989, p. 17), "zero one" or "01" (Bishop et al., 2011, p. 352), "something one" or "S1" (Bishop et al., 2011, p. 353), "minus one" (Aze, 1989, p. 17; Liebeck, 1990, p. 226) or "m1" (Liebeck, 1990,p. 226) or "negative one" (Bishop et al., 2011, p. 354). Yet, making sense of formal integer notation is difficult for students. With the introduction of negative numbers, the "-" symbol takes on three meanings: unary (negative), binary (subtraction), and symmetric (opposite)—see Table 1 for examples (Barber, 1925, 1926; Gallardo & Rojano, 1994). In this article, I use the term "minus" generically to refer to the "-" symbol, "subtraction sign" to refer to the binary meaning of the minus sign, and "negative sign" to refer to the unary meaning of the minus sign.

Table 1

Three Meanings of the Minus Sign

Meaning of the Minus Sign	**Explanation**	**Example**
Binary Function	Subtraction	9 − 3 (operation)
Unary Function	Negative Number	-7 (signed number)
Symmetric Function	Taking the Opposite	-(4 + 2) = -(6) = -6

Before instruction on negative numbers, students primarily rely on the binary meaning of the minus sign, interpreting all minus signs as subtraction signs (Bofferding, 2010); this interpretation may be one reason that students call the negative sign a “dash” or “minus.” However, students need to understand that the “dash” or negative sign serves to designate numbers that are ordered before zero and have different values than positive numbers.[1] If they continue to treat negative signs as subtraction signs, students may interpret negative numbers as incomplete subtraction problems; for example, some students have interpreted -5 as 5 − 5 (Bofferding, 2010; Hughes, 1986). Especially for problems with multiple minus signs (e.g., 3 - - 5), students who only have experience with the binary meaning of the minus sign might ignore the negative signs completely because they do not fit their ideas of subtraction or they might subtract twice when two minus signs are next to each other (Bofferding, 2010; Murray, 1985). In terms of the binary meaning of the minus sign, students must also make sense of the non-intuitive result of subtraction making larger when subtracting a negative number (Bruno & Martinón, 1999). If students do not adequately understand the three meanings of the minus sign, further difficulties arise when they attempt to simplify and solve algebraic expressions (Lamb et al., 2012; Vlassis, 2004, 2008).

Building Mental Models of Integers

As students make sense of integers as directed magnitude and interpret integer order, values, and notation, they must also make sense of the relationships among these concepts. In their interviews, Peled et al. (1989) found that students who ordered negatives before zero and symmetric to the positive numbers, primarily talked about adding and subtracting the integers abstractly, counting up or down the number sequence rather than reasoning about assets and debts. Based on these results and drawing on the work of Resnick (1983), Peled et al. (1989) concluded that these students relied on a number line mental model to support their thinking

[1] Throughout the study, I presented negative signs as smaller than minus signs and slightly raised.

about negative integers. They found that students either used a Divided Number Line model, where students calculate to and from the zero point and interpret the positive and negative halves as separate, or a Continuous Number Line model, allowing students to move easily between positive and negative numbers. Peled (1991) later expounded that students' integer descriptions are consistent with them having one or two types of mental models: one regarding quantities, where negative numbers are considered to be unfavorable amounts, such as debts, where the larger the unfavorable amount, the smaller the number and one regarding a number line where negative numbers are abstract entities ordered to the left of zero, with larger values ordered further to the right.

For both mental models, Peled (1991) described four levels of understanding: students at the first level of integer knowledge know the order of all integers, with larger numbers further to the right on the number line and the numerals ordered symmetrically around zero; at the second level, students can add positive numbers to any integer; at the third level, students can add or subtract two positive or two negative numbers; and at the fourth level, students can add or subtract any two integers. Although Peled theorized about *what* students should be able to do at each level, she did not specify *how* students develop mental models of integer order and values at the first level of integer knowledge. Further, she did not specifically describe the role of directed magnitude understanding, and even though Peled et al. (1989) acknowledged that notation contributes to students' difficulties in working with integers, the roles of notation and the multiple meanings of the minus sign within the mental models are not clear.

The extant literature on integer instruction provides little insight into how students develop mental models of integer order and values, directed magnitude, or integer notation. Most studies on instruction focus on integer computation, addressing the meaning of the minus sign implicitly. The two primary models for teaching negative numbers are the cancellation model and the number line model.

The cancellation model involves using the additive inverse principle, which could aid partial understanding of the unary minus sign. However, the model does not emphasize order, and in some cases, subtracting involves both addition and subtraction, potentially confusing the binary meaning of the minus sign (e.g., for the problem 5 - -3, students are told to start with five positive chips, *add* three positive and three negative chips—or zero pairs—then *subtract* the three negative chips, leaving eight positive chips). However, students can also learn about compensation, that adding a negative is equivalent to taking away a positive (Liebeck, 1990; Williams, Linchevski, & Kutscher, 2008), which can support their understanding of directed magnitudes and the binary meaning of the minus sign in relation to subtracting negative numbers.

The number line model highlights the order of negative numbers compared to positive numbers, and number values can be interpreted as distances from zero in opposite directions, providing meaning for the unary meaning of the minus sign and directed magnitudes. Fischbein (1977) and Freudenthal (1973) agreed that number line models could be helpful for supporting integer addition, and the National Council of Teachers of Mathematics (2000) recommends students use a number line model to explore numbers less than zero. Further, Hativa and Cohen (1995) found that fourth graders could successfully compare integers after completing an intervention where they made computations to reach a target integer and received visual feedback of their calculations on a number line model. Seventh graders who participated in instruction on net worths that incorporated empty number lines used magnitude reasoning in a productive way; they determined that higher or larger negatives are further away from zero and used this reasoning to determine that -$20,000 represents a net worth greater than -$22,000 (Stephan & Akyuz, 2012). However, some instructional strategies for adding and subtracting using the number line may send conflicting messages about the meanings of the minus sign. Common rules for subtraction often involve interpreting the unary meaning of the minus sign as a cue to “face in the *negative* direction” (Liebeck, 1990, p. 233) on the number line, treating the negative sign as an operation and confusing it with the opposite meaning of the minus sign. Other rules have included interpreting the binary meaning of the minus sign (subtraction) as a cue to face left, while the negative means to move backward on the number line (Moreno & Mayer, 1999). Across these forms of instruction, there is varying emphasis on the multiple meanings of the minus sign, making it difficult to discern their impact on students’ understanding of integers.

The Present Study

Drawing on the body of research presented above, I conjecture that instruction designed to highlight multiple meanings of the minus sign could support students’ developing mental models of negative numbers. One possibility is that students must learn the negative meaning of the minus sign as well as the order and values of the negative numbers (*Unary Instruction*) through targeted experiences. For instance, students might need instruction that numbers further to the right on the number line correspond to larger values, regardless of whether they are positive or negative. With this understanding, it is possible that they could extrapolate how to add or subtract two negative numbers based on adding or subtracting two positive numbers.

A second possibility is that students only need informal exposure to negative integers. Rather, they may need targeted experiences with the binary meaning of addition and subtraction (*Binary Instruction)* within a directed magnitude context because elementary students tend to use the descriptions of *plus*, *more*, and *higher* as well as *less*, *down*, and *lower* inconsistently when talking about adding or subtracting negative numbers, respectively (Bofferding, 2010). By moving *more* or *less* distance from zero and other points along a number line in either the *positive* or *negative* direction, they may learn that numbers are symmetrical around zero and that "negative" refers to numbers to the left of zero (opposite the positive numbers). A third possibility is that students need instruction on both meanings of the minus sign (*Combined Instruction*) in order to develop formal mental models of integers.

The purpose of the research reported in this article is to detail first graders' mental models of negative integers and to investigate how instruction on the binary and/or unary meanings of the minus sign supports or constrains their thinking about negative integers. The research questions I explored are as follows:

1) What are first graders' mental models of integers:
    a) For integer order and values? and b) For directed magnitudes?
2) How do first graders' mental models of integers change depending on whether they receive instruction on the unary and/or binary meaning(s) of the minus sign?

Theoretical Perspectives

Two major theories influenced the design of the study and analysis of the results: Vosniadou's (1994) theory of conceptual change and Case's (1996) theory of *central conceptual structures for number* (CCSN). There are a few prominent theories of whole number and operations development (e.g., conceptions of quantities theory—Fuson, 1992a, 1992b—and counting types theory—Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983) that detail the construction of units and focus on quantity conceptions of number. However, I have drawn on Case's (1996) theoretical framework because it discusses the relationships among positive number order, values, and operations in terms of a mental number line. In concert with Peled's (1991) notion of a number line mental model for integers, I applied Case's framework more broadly to include the domain of negative integers and used it to inform the selection of the assessment and instructional tasks used in this study. Vosniadou and Brewer (1992) provide a method for characterizing domain-specific differences and changes in students' mental models, which supported the data analysis. Further, both theories emphasize the importance of instruction on students' developing understanding of new concepts, an important part of the research design used for this study.

Framework Theory of Number

According to Vosniadou and Brewer (1992), students develop an *initial mental model* for concepts based on direct experiences with the world, and they generate a set of rules, called a *framework theory,* based on those experiences. Regarding number concepts, children develop their framework theory based on observable quantities (words in brackets are my addition):

- Numbers are *counting numbers*
- Numbers are discrete: There is no other number between a number and its next
- There is a smallest number (0 or 1)
- Numbers can be ordered by means of their position in the count list [so that numbers farther from zero are larger]
- "Longer" numbers (i.e. with more digits) are bigger
- Addition … "make[s] bigger"
- Subtraction … "make[s] smaller"
- Every number has only one symbolic representation

  (Vosniadou, Vamvakoussi, & Skopeliti, 2008, p. 11).

As indicated by Ashvin in the opening quote, students use this framework theory successfully for many years before they encounter negative numbers. Other researchers have also identified one or more of these principles as important in students' reasoning about numbers (see also Dehaene, 1997; Fischbein, Deri, Nello, & Marino, 1985; Gelman, 2000, Hiebert, 1992; Moskal & Magone, 2000; Resnick, 1992; Stavy, Tsamir, & Tirosh, 2002). More specifically, Case (1996), provides a helpful account for thinking about how students' understanding of these principles develops in relation to a number line mental model for whole numbers.

Central Conceptual Structure of Number

According to Case's (1996) theory for CCSN, before children develop a central conceptual structure for whole numbers, they are considered pre-dimensional. Then, once they have developed a CCSN, they pass through three levels of thinking in which they can use their CCSN with increasing degrees of complexity: the uni-dimensional level, bi-dimensional level, and integrated bi-dimensional level (Okamoto & Case, 1996).

Pre-dimensional level. Based on prior research, Case (1996) posited that children ages 3–5 respond to number questions using one of two separate cognitive structures at a time: a structure for enumerating and a structure for judging global quantity. If asked to determine how many objects are in a small set, children reasoning at the pre-dimensional level can use their enumeration structure to count the objects and identify the last word in the count as the cardinality of the set of objects or identify if someone has failed to complete this process correctly (Gelman, 2000). However, when recreating small sets, or determining which of two small sets of objects has more, the same children will use their global quantity cognitive structure to perceptually determine the answer (Sophian, 1987; Starkey, 1992).

At the pre-dimensional level, although children count to determine the quantity of *one* set, they will not use this process to determine which of *two* sets of objects has more (Fuson, 1988). Instead, they try to determine which set has more visually (Case, 1996; Sophian, 1987), even if the difference between sets is not visually obvious (Griffin, 2005). If these children see an adult count two sets, they can determine which has more, suggesting that they understand the cardinality principle but have memory load limitations that make this process too difficult for them to do independently (Curtis, Okamoto, & Weckbacher, 2009). Further, they will not consistently determine which verbally-presented, single digit number is greater than another even though they can count to those numbers and determine which set containing those amounts is greater (Siegler & Robinson, 1982)—evidence that the two cognitive structures are not coordinated at this point but are used separately.

Uni-dimensional level. Eventually, around the ages of 5–7, children coordinate their enumeration and global quantity cognitive structures. The result is a new cognitive structure: the CCSN (Case, 1996) or *mental number line*, as originally described by Resnick (1983)[2]. Case and colleagues hypothesize that the CCSN underlies numerical thinking and subsumes the previous two, separate cognitive structures (Case, 1996; Griffin, Case, & Capodilupo, 1995). The CCSN, or knowledge network, consists of several numerical concepts and the relations among them: knowledge of the verbal number sequence, a tagging routine for keeping track of a count, knowledge of number quantities, and knowledge of symbolic representations for number and operations (including the binary meaning of the minus sign), which is mapped onto the other three components. According to Case (1996), a central conceptual structure "forms a sort of lens through which children view the world. It also constitutes a tool they use to create new

[2] Fuson (1992b) argued that *mental number list* is a more accurate description because the conceptual structure is based on the number sequence; whereas, number lines are measurement models. However, I use *number line* in accordance with Resnick (1983) to stay in line with the theory as described by Case (1996).

knowledge" (p. 8). With the development of the CCSN, then, children can solve more complex problems than they previously could.

Upon the initial formation of their CCSN, students are considered at the uni-dimensional level because they are only able to use a single mental number line to solve numerical questions involving one dimension, such as adding ones. At the uni-dimensional level, they can determine that larger quantities, or *values* as I refer to them[3], correspond to more motor movements in the tagging process (which are eventually done mentally and then not at all), to number names further up in the naming sequence, and to larger numerals, which are written further to the right on a number line. By counting up and down their mental number line, students can solve single-digit addition and subtraction problems, where addition (represented by the plus sign) results in *more* and subtraction (represented by the subtraction sign) results in *less,* and make judgments about the relative magnitudes of two single-digit numbers (Case, 1996; Griffin et al., 1995; Siegler & Ramani, 2008, 2009).

Bi-dimensional levels. Around 8 years of age, with corresponding increases in working memory capacity, Case (1996) suggested that students can coordinate two mental number lines simultaneously. Therefore, at this bi-dimensional level they can operate with two dimensions. For example, when finding the difference between two numbers, students can use one mental number line to keep track of the start and ending points and a second mental number line to keep track of the count (Okamoto & Case, 1996). Finally, around 10 years of age, students are expected to develop an integrated bi-dimensional understanding of number. Therefore, they

[3]For children, quantities are represented with countable objects (e.g., 3 blocks, 4 fingers). Sometimes negative amounts are represented using colored chips: 4 red chips to represent "-4" and 4 yellow chips to represent "4". "Quantity" in reference to negative amounts in this situation might be confused with absolute value rather than the relationship that $-4 < 4$. Therefore, I use the term "value" instead of "quantity" to avoid this ambiguity.

should be able to apply their understanding of two-digit numbers to the larger number system and also reason about the relations between two sets of numbers (Case, 1996).

Drawing on the theory. The CCSN theory provides an important theoretical perspective for this study because it helps explain how students' understanding of various number concepts, such as order, value, operations, and notation, develop and become coordinated (Case, 1996). Because negative integers follow the rules that the CCSN theory hypothesizes students should understand at the uni-dimensional level (i.e., numbers further to the right on the number line are greater, numbers further to the left on the number line are less, and numbers can be ordered in relation to each other), students exhibiting reasoning at the uni-dimensional level should also be able to order and determine the relative value of negative integers if they have appropriate instructional experiences. However, unlike with the whole numbers where movements to the right on the number line correspond to numbers or quantities that are *more* in the positive direction, students need to understand that they can also get *more* in the negative direction (and less in either direction as well). Therefore, as they make the transition to incorporating negative numbers on their number line mental models, they need to understand the new order and values of the negative numbers as well as directed magnitudes.

Mental Models and Conceptual Change

Vosniadou and Brewer (1992) describe a conceptual change framework that can clarify how students might move from using a mental number line for positive integers as described by Case (1996) to a mental number line for both positive and negative integers. Conceptual change involves enrichment or revision of students' current conceptual structures to accommodate new knowledge (Vosniadou, 1994). During the conceptual change process, students frequently develop alternative conceptions to the culturally-defined formal model (which is considered the end goal), and their attempts to incorporate the new information into their conceptual structure provide insight into how they are thinking about these concepts (Stafylidou & Vosniadou, 2004). Vosniadou and Brewer (1992) identified three categories useful for explaining students' mental models of a concept: initial, synthetic, and formal. Below, I explain these categories by providing examples that integrate the CCSN theory and the negative number literature for the two constructs under investigation in this study: integer order and values and directed magnitude.

Initial Mental Models

Students' initial mental models for integers draw on their framework theory for whole numbers (Vosniadou & Brewer, 1992). Therefore, students with initial mental models for numbers might answer questions about negative integers using positive integer rules. Students who rely on a CCSN and associate the minus sign only with subtraction might treat negative integers as if they were positive because the binary meaning of the minus sign does not carry any meaning for them in a unary context. For example, students might ignore the negative signs and order negative numbers as if they were positive, which could explain the results Peled et al. (1989) found. Alternatively, if they do not ignore the minus sign, students might rely on its binary, subtraction meaning and interpret -5 as $5 - 5$, (Hughes, 1986; Schwarz et al., 1993).

With the introduction of negative numbers, students may classify numbers greater than zero as high (or positive) and less than zero as low (or negative). Students who rely on a positive-number-only framework theory for operations equate addition with "more," moving right on the number line, or going "higher" in the number sequence and subtraction with "less," moving left on the number line, or going "lower" in the number sequence. Therefore, when interpreting movements "more negative" or "less negative," they might ignore the direction of "positive" or "negative" and move *more* or *less* based on the positive number line, or they might pay attention to moving in the positive or negative direction without paying attention to whether they should move "more" or "less" in that direction.

Synthetic Mental Models

In some cases, when students learn about a new concept, they begin to reorganize their conceptual structure and, in this case, application of number principles in an effort to assimilate the new information while maintaining their current knowledge structure. The result is a *synthetic mental model* because it is a synthesis of multiple ideas. Students often develop several synthetic mental models and, because they are relatively unstable, may shift among them (Vosniadou, 2007).

When students learn that negative integers exist, they might accept that negative integers are less than zero but argue that -5 is greater than -3 (Schwarz et al., 1993) because -5 is further away from zero than -3 is, just as 5 is further away from zero than 3 is (Stavy et al., 2002). This synthetic mental model maintains the principle that numbers further in the counting sequence from zero have greater values while allowing for numbers smaller than zero to exist. Similar to their interpretations of integer values, students with synthetic mental models for operations with directed magnitudes begin to coordinate "more" and "less" with directions such as positive/negative or high/low. However, they may switch between focusing on either element inconsistently.

Formal Mental Models

When students successfully reorganize their framework theory and accommodate the new information to reflect adult understanding—as aligned with sanctioned mathematics in the culture—the result is a *formal mental model*. Ultimately, students need to coordinate their order and value mental models and directed magnitude mental models to reason about quantities from both positive and negative perspectives.

A formal mental model of integer values and order extends the number sequence from the CCSN, so that the numerals are symmetric around zero, with a negative sign (unary meaning of the minus sign) preceding numerals ordered to the left of zero (less than zero). Additionally, numbers decrease in value further to the left on the mental number line and increase in value further to the right. Furthermore, students with formal mental models of directed magnitudes understand that movements in a more negative direction (to the left) or toward decreasing values (Mukhopadhyay, 1997) on a mental or physical number line correspond to movements in a less positive direction.

Methods

Participants

This study took place at a K-5 school in California during the last two months of the school year. The school's three predominant ethnic groups were Asian (34.5%), Hispanic or Latino (28.1%), and White (24.8%); 47.2% were English Language Learners, and 29.6% qualified for free and reduced lunch (California Department of Education, 2010). I chose to work with first graders because students ages 6-8 typically function at the uni-dimensional level (Case, 1996; Okamoto & Case, 1996). Given the study was conducted at the end of the school year, the students had received instruction on adding and subtracting positive, single-digit numbers. Although students at this level are less likely to have formal knowledge of integers, I reasoned that they should be able to order and determine the relative value of negative integers if they have sufficient instructional experiences.

All 79 first grade students (from the four first-grade classrooms) were recruited. Sixty-three students returned permission slips (there was no obvious pattern in the students who did not return permission slips), but two of them could not complete an initial interview in English and were not included in the study. The sixty-one first graders who were included in the study averaged 6 years, 10 months in age (*SD* = 4 months) at the beginning of the study.

Study Design

The study consisted of a pretest, instructional intervention, and a posttest. Both pre- and post-instructional tests were conducted as individual interviews with the students. A trained graduate student and I served as interviewers.

Pre- and post-instructional tests. The same test (with modifications described below) was administered to students before and after the instructional intervention. It consisted of a series of questions designed to assess students' understanding of the components underlying the CCSN at the uni-dimensional level (Case, 1996) as they related to negative numbers. The test questions focused on (a) names and notation of integers, (b) integer order, (c) integer values, (d) directed magnitude, and (e) operations with integers, including understanding of commutativity (see Table 2 for example questions and Appendix A for the interview protocol).[4] Most of the questions were modeled after those from other studies of students' negative or whole number understanding. For example, saying the forward number sequence is a prerequisite skill for developing a CCSN (Griffin, 2005); therefore, I included a question to determine if students could count below zero. I used Cronbach's alpha to determine the internal consistency reliability

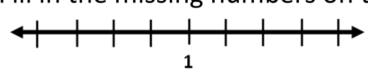
[4] The data reported in this article do not include students' responses to the questions pertaining to operations. Information about that aspect of the study is reported in Bofferding (2011a, 2011b).

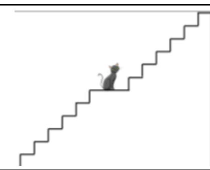
of the test items (Crocker & Algina, 2006). Both pretest and posttest measures were highly reliable (α = .90 and .92, respectively).

The tests were administered to students as individual interviews. Each student was interviewed outside of the classroom in a nearby multipurpose room using a set protocol. The student sat at a table facing the wall, and the interviewer sat next to the student. We positioned a camera to capture the student's gestures and work and placed an additional voice recorder near the student to pick up any audio not captured on the camera. The student worked out of a booklet, which had one problem printed on each page. After the student solved a problem, the interviewer asked, "How did you solve it?" to learn more about his/her strategies (Siegler, 1996). We did not give feedback on correctness but provided generic encouragement (e.g., "Great!"). For the post-instructional interview, I randomly assigned students from each instructional group to an interview order to control for students from one group having greater or fewer days elapse between their instruction and posttest interview.

Table 2

Example Pre- and Post-Test Interview Questions and References From Which the Questions Were Modified

Order and Value Questions		
Counting Backward (n = 1)	Griffin (2005)	Start at five and count backwards as far as you can. Can you count back any further? Is there anything less than <number student said>?
Filling in a Number Line (n = 1)		Fill in the missing numbers on the number line. 
Ordering Integers (n = 2)	Schwartz, Kohn, and Resnick (1993)	Put these number cards in order from the least to the greatest: $\boxed{2}$, $\boxed{-3}$, $\boxed{0}$, $\boxed{-9}$, $\boxed{3}$, $\boxed{8}$, $\boxed{-5}$
Greatest/ Least (n = 4)	Schwartz et al. (1993)	After ordering the integer cards: Which is greatest? Which is least?

Greater Integer (n = 7)	Peled et al. (1989)	What are these two numbers? $\boxed{3}$ $\boxed{-9}$ Circle the one that is greater. Story problem version: Two children are playing a game and trying to get the highest score. Who is winning?
Numbers (n = 2)	Fagnant, Vlassis, and Crahay (2005)	Circle all of the numbers in the equation. Read the numbers you circled. How did you decide what to circle? 6 + -2 − 7 = -3
Directed Magnitude Questions		
High, Low (n = 4)		Put an X on the stair where the cat will be if she moves 1 stair more low. How did you solve it? 
Positive, Negative (n = 4)		(Given on posttest only) Put your finger at 0. Move your finger more negative 4 tick marks on a number line and draw a square.
Operations Questions		
Operations (n = 2)		What does it mean to add? (What happens when you add?)
Signs (n = 2)	Fagnant et al. (2005)	Circle all of the plus signs and minus signs, which tell you whether to add or subtract. How did you decide what to circle? -4 − 3 + -1 = -8
Commutative? (n = 6)	Baroody and Gannon (1984)	Look at the problems. You don't have to solve them, but will these problems give you the same answer? How do you know? 3 − 1 and 1 − 3
Bi-dimensional Subtraction (n = 2)	Okamoto and Case (1996)	23 Go ahead and solve this problem. − 15 How did you solve it?
Addition and Subtraction (n = 26)	Peled et al. (1989)	S = smaller absolute value; L = larger absolute value; L, S, X > 0 Solve single-digit problems of the following forms: L − -S; S − -L; S − L; -S − L; -S + L; -L + S; -S − -L; -L − -S; -X − -X; L + -S

Integer order and value questions.

Counting backward. The counting task probed students' knowledge of the decreasing verbal naming sequence, a foundational part of the CCSN, which students could use to indicate knowledge of negatives. On both the pre- and post-instructional interviews, students were asked to start at "5" and count backward as far as they could. If they stopped at 1 or 0 they were asked, “Can you count back any further? Is there anything less than <last number said>?” This question was asked prior to them seeing a negative integer to prevent the structure of the interview influencing their answer.

Filling in a number line. Having students fill in a nearly blank number line was another way to determine how they order integers and with what notation. For both interviews, students were given a horizontal number line with “1” labeled and were asked to fill in the missing numbers. Once again, to see what notation students would use, they answered this question before seeing negatives printed on later questions. If they left any spaces empty, they were asked if any numbers went there.

Ordering integers and greatest/least. The ordering task provided additional information on how students interpret and order integers. If students did not include negatives on their number lines, this task would clarify whether they knew about negatives. Twice, students received a set of cards spread out in front of them to order from least to greatest. This task also probed how students coordinated their integer value judgments with integer order because they then indicated which numbers were the least and greatest.

Greater integer. These questions targeted students’ understanding of integer values. Students compared seven pairs of numbers (one positive-positive, three negative-negative, three positive-negative) in isolation or in a story context and circled that one that was greater. In the positive-negative case, the absolute value of the negative integer was always greater than the positive number (e.g., -7 vs. 3).

Numbers. These questions were designed to elicit students’ understanding of the unary meaning of the minus sign (e.g., signed numbers). However, students’ responses were inconsistent and unrelated to their other answers (e.g., students who used the term negative to refer to negative integers in other problems did not circle the negative signs, and others circled the negative sign due to imprecision in their circling, ignoring the negative sign when naming them). Therefore, these two questions were not included in further analysis.

Directed magnitude mental model questions.

High, Low. The directed magnitude questions probed how students interpret moving *more* or *less* in a given direction. Students had to determine the movements of a cat when it moved more high, less high, more low, and less low.

Positive, Negative. On the posttest, students answered an additional four questions with the language of moving more positive, less positive, more negative, and less negative on a number line; the positive/negative versions of the questions were not asked on the pretest to avoid giving students exposure to this terminology prior to the intervention.

Random assignment to instructional groups. I used two factors to stratify students before random assignment to instructional groups. First, classroom teachers rated their students' overall performance in number and operations, which they referred to as "number sense" (low, medium, high). Second, I scored students' integer order and value knowledge on the pretest (low = 1, low–medium = 2, medium = 3, medium–high = 4, and high = 5) for three tasks: filling in the number line, ordering the integers, and determining which integer was larger. Low responses involved treating the numbers as positive, medium responses involved recognition of negative numbers as different than positive numbers with some answers correct, and high responses involved correct answers. I averaged their scores on the three tasks to determine their integer knowledge scores (see Bofferding, 2011a for more detail on this process) and then combined these with the teachers' ratings to form four categories of students: (a) high number sense, middle–high integer knowledge; (b) high number sense, middle–low integer knowledge; (c) medium number sense, middle–low integer knowledge; and (d) low number sense, low integer knowledge.

I randomly assigned students from each category to one of three instructional groups: Unary Instruction, Binary Instruction, or Combined Integer Instruction (described in the Instructional Groups section). This process ensured that each instructional group had similar numbers of children from each category. Due to scheduling conflicts, I had to move two students to different groups after assignment, but I switched students from the same category with similar integer knowledge scores to balance the instructional treatment groups. After random assignment, each group had a similar mix of female and male students, teacher-designated number sense levels, and integer knowledge levels, as shown in Table 3.

Table 3

Composition of the Three Instructional Groups After Randomization

	Unary Instruction	Binary Instruction	Combined Instruction
Number of Males	10	10	10
Number of Females	10	11	10
(Teacher) High; (Pretest) High	3	3	3
(Teacher) High; (Pretest) Med-Low	5	6	4
(Teacher) Medium; (Pretest) Med-Low	8	7	9
(Teacher) Low; (Pretest) Low	4	5	4
Average Integer Arithmetic Score	4.2	3.9	4.3
(Median Score)	3.0	3.0	3.5

Instructional intervention. According to Vosniadou (1994), one way to encourage conceptual change is by providing students with instruction and exposure to new concepts. Across all three instructional groups, the general format for instruction involved a short, whole-group introduction and discussion of a particular concept followed by a partner exploration or game involving the concept. The initial lessons for each topic focused on general principles (e.g., the same symbols can mean different things, order sometimes matters), which were meant to connect to students' previous knowledge so that they could relate the principles to subsequent integer instruction (Bruner, 1960/2003). Subsequent lessons focused on the specific integer concepts (e.g., the minus sign can indicate subtraction or a negative number, subtraction is not commutative). Recent research (e.g., Ramani & Siegler, 2008) and curricula (Griffin, 2004) demonstrating the successful use of linear, numerical board games—where squares are arranged in a line and labeled with sequential integers and children count as they move spaces—to promote early mathematical learning inspired the use of games and explorations involving number lines[5] as the main part of the instructional activities across all groups (see Bofferding, 2011a for more description on the instructional design).

During the whole class segments, I asked students questions related to the main activities, had students provide explanations, and supplemented with my own explanations. During the exploration time, I helped partners and asked them questions to check their understanding about the mathematics in the activities. I provided each instructional group with eight, 45-minute instructional sessions over two weeks. At their group's instructional intervention time, students left their classrooms and gathered together in the multipurpose room. After one group had instruction, they returned to their respective classrooms, and the next group met. The groups met for instruction at different times over the week and in different orders to control for effects of time of day on learning. I taught all three instructional groups and followed strict lesson plans to ensure that any overlapping instruction among groups was similar.

[5] Although Fuson (1992b) argued against the use of number lines with young students, the Number Worlds curriculum (Griffin, 2007), which is based on the CCSN theory, uses number lines and is an effective curriculum (Griffin, 2004). I chose to use number lines as well. The students in the sample had previous exposure to using number lines for ordering positive numbers in their mathematics curriculum.

Instructional groups. In an attempt to investigate the role of students' understanding of the two meanings of the minus sign (unary and binary) on their developing understanding of negative numbers, I created three instructional groups: the Unary group, the Binary group, and the Combined Integer Instruction group. The use of number lines helped illustrate the unary and/or binary meanings of the minus sign as described next.

Unary instruction group. The *Unary Instruction group* explored the order, values, and symbols corresponding to negative integers. Rather than adding and subtracting integers, they investigated similarities and differences between the appearance and use of subtraction and negative signs by comparing and contrasting mathematical expressions (e.g., What is the same and different about these two expressions? -4 − 5 versus -4 − -5), identifying when the sign represented subtraction and when the sign indicated a negative number (the binary and unary meanings of the minus signs, respectively), and playing a matching game with positive and negative integers for extra practice. They also played two ordering games: one where they ordered integers on a vertical number line and determined which of two integers was greater (based on which integer was higher—located closer to the surface if both were below water or

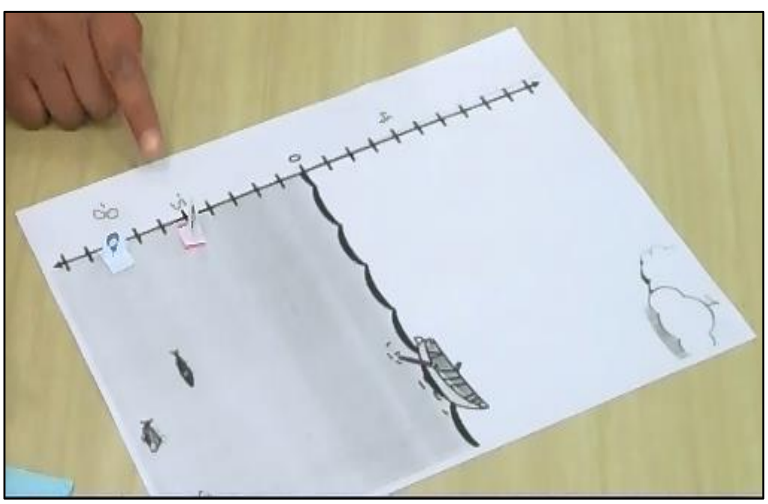


Figure 1. Students draw number cards, label the numbers on the game board number line, and then use the context to determine which of two integers is greater (in this case, -5 versus -8).

Figure 2. Students try to collect cards to make three in a row (e.g., -3, -4, -5) and check the order using the number line.

located further out of the water if one or both were above water—see Figure 1) and one where they tried to collect cards that would fill in three numbers in a row on their vertical number line (see Figure 2). Playing the games over multiple days gave them extra practice with the concepts. The focus on integer order and relating integer values to their order on the number line was meant to help highlight the unary meaning of the minus sign and the difference between positive and negative numbers. This group did not participate in any instructional activities related to directed magnitudes or adding and subtracting negative integers.

Binary instruction group. The *Binary Instruction group* explored directed magnitudes and addition and subtraction concepts by moving forward and backward on the number line. In their first set of games, students moved an elevator up and down on a vertical number line (with integers indicating floors above and below ground) to model adding and subtracting positive numbers (see Figure 3). As they played the game, they explored problems where the numbers were switched (e.g., $5 - 3$ versus $3 - 5$). Students could see the order of the integers but were not asked to discuss the order of the integers or to identify which integers were greater or less than the others as in the other groups, and I referred to -3 as “three floors below zero” or “three floors underground.” I did, however, make sure students used the negative sign when necessary, emphasizing that 3 and -3 referred to different floors. Students played the game on a second day for extra practice.

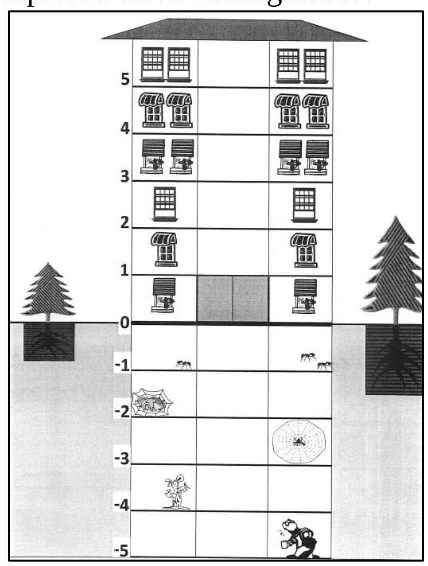


Figure 3. In this game, students moved an elevator game piece up and down the floors to solve problems where they added or subtracted positive integers.

Because addition and subtraction take on new meaning with negative integers (addition can make numbers smaller if you add a negative), the second part of instruction focused on directed magnitudes. Rather than using the common number line rules for integer addition and subtraction where addition means “move forward,” subtraction means “move backward,” and the negative sign is a cue to “face in the negative direction,” students played games on a horizontal number line where they moved in directions that were more negative or less positive (to the left) or more positive or less negative (to the right) depending on whether they were adding or subtracting a positive or negative number with the goal of getting to “positive park.” In another variant of the game, students tried to get to “negative nettles” (see Figure 4). The game cards included the names and symbols for the integers, so the students were exposed to these concepts. However, the number line was only labeled with zero, and students did not receive instruction on the specific order and values of the integers (other than the general ideas of more or less positive or negative).

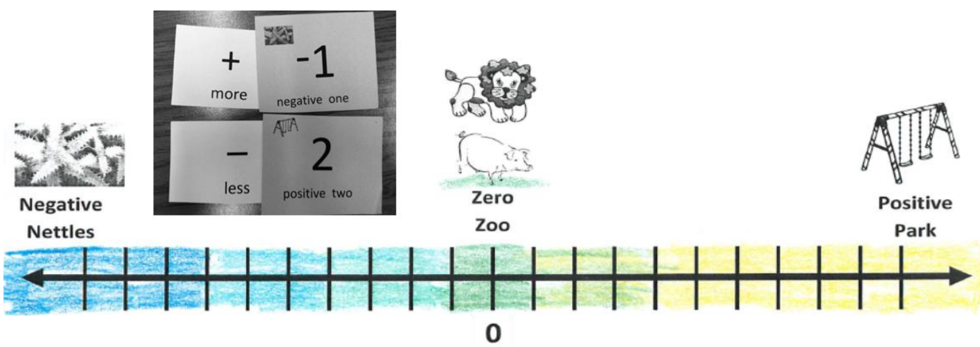


Figure 4. Students moved their game piece along the number line to try and get to positive park. They drew an operation card and integer card. In the above examples, either group of cards would result in the student moving to the left, toward negative nettles. Students in the Binary Instruction group only saw “0” labeled on the number line; whereas, students in the Combined Instruction group saw all integers labeled.

Combined integer instruction group. The *Combined Integer Instruction group* had a combination of instruction from the other two groups; they had three lessons in common with the Unary Instruction group and then five lessons in common with the Binary Instruction group. Therefore, they received instruction and played the games related to identifying negative numbers, ordering them, and determining which is greater—as in the Unary Instruction group—and playing the elevator and other operations games to practice adding and subtracting integers with directed magnitudes—as in the Binary Instruction group. They did not do the extra practice activities (designated as "review" in Table 4)[6]. Table 4 summarizes the instruction for each group and shows the overlapping lessons between groups. Complete lesson plans are available in Bofferding (2011a).

[6] There was one exception. On Day 2 of Week 1, the Binary Instruction group completed an activity where they moved an elevator on a game board and recorded their movements in terms of arithmetic problems (e.g., starting at floor 2 and going down 1 floor would be 2 − 1 = 1). Recording the problems was difficult for them, so on the review day I gave them specific problems to explore using the elevator context. Therefore, when it was time for the Combined Integer Instruction group to do the elevator activity, I had them do the activity from the second day as opposed to the initial version.

Table 4

Lesson Topics and Explorations for Each Instructional Group With Common Lessons Between Groups Indicated by Topic Letters

Group	Day 1 (Week 1)	Day 2 (Week 1)	Day 3 (Week 1)	Day 4 (Week 1)
Unary Instruction (n = 20)	**A) Topic:** symbols can have more than one meaning depending on their context and how they are arranged **Explore:** use two lines to make as many symbols as you can; which ones look the same but mean different things?	**B) Topic:** minus signs can have different meanings; how to identify negative numbers **Explore:** compare and contrast problems with minus signs and/or negative signs; sort problems	**Topic:** (review) distinguish between minus and negative sign **Explore:** sort problems; Memory game where they match integers (negative to negative, positive to positive)	**C) Topic:** integers can be ordered on a vertical number line (distances above and below water); greater integers are ones further out of the water **Explore:** move your game piece above or below water according to your card; who has the greater number?
Combined Integer Instruction (n = 20)	**A) Topic:** symbols can have more than one meaning depending on their context and how they are arranged **Explore:** use two lines to make as many symbols as you can; which ones look the same but mean different things?	**B) Topic:** minus signs can have different meanings; how to identify negative numbers **Explore:** compare and contrast problems with minus sign and/or negative sign; sort problems	**C) Topic:** integers can be ordered on a vertical number line (distances above and below water); greater integers are ones further out of the water **Explore:** move your game piece above or below water according to the integer card; who has the greater number?	**D) Topic:** changing the order of events (e.g., sit, stand versus stand, sit) will sometimes give us a different end result; changing the order of numbers in a problem will sometimes give us a different result **Explore:** share 4 cakes with 2 people and 2 cakes with 4 people
Binary Instruction (n = 21)	**D) Topic:** changing the order of events (e.g., sit, stand versus stand, sit) will sometimes give us a different end result; changing the order of numbers in a problem will sometimes give us a different result **Explore:** share 4 cakes with 2 people and 2 cakes with 4 people, discuss the result	**Topic:** order does not matter in addition but does matter in subtraction **Explore:** move your elevator above or below ground according to the operation and integer cards; record the arithmetic problem	**E) Topic:** (review) order does not matter in addition but does matter in subtraction **Explore:** move your elevator to solve addition (e.g., 2 + 3, 3 + 2) and subtraction problems (e.g., 3 − 1, 1 − 3); what do you notice about the answers?	**F) Topic:** directed opposites: when dealing with opposites, more of one thing is less of the other (on a continuum, more in one direction is less in the other) **Explore:** use blue and yellow food colors to mix colors along a continuum (blue, green, yellow); adding yellow makes water more yellow, adding blue to yellow water makes it less yellow (to green then to more blue)

Table 4 (cont.)

Group	Day 5 (Week 2)	Day 6 (Week 2)	Day 7 (Week 2)	Day 8 (Week 2)
Integer Properties (n = 20)	**Topic:** (review) integers can be ordered on a vertical number line (distances above and below water); greater integers are ones further out of the water **Explore:** move your game piece above or below water according to your card; who has the greater number?	**Topic:** (review) determine relative values of integers **Explore:** play card game War: Which integer is greater? Variation: which integer is least?	**Topic:** (review) determine relative order of integers **Explore:** play to get three or four consecutive integer cards in a row (check the order using the vertical number line)	**Topic:** (review) determine relative order of integers **Explore:** play to get three or four consecutive integer cards in a row (check the order using the vertical number line)
Combined Integer Instruction (n = 20)	**E) Topic:** order does not matter in addition but does matter in subtraction **Explore:** move your elevator to solve addition (e.g., 2 + 3, 3 + 2) and subtraction problems (e.g., 3 − 1, 1 − 3); what do you notice about the answers?	**F) Topic:** directed opposites: when dealing with opposites, more of one thing is less of the other (on a continuum, more in one direction is less in the other) **Explore:** use blue and yellow food colors to mix colors along a continuum (blue, green, yellow); adding yellow makes water more yellow, adding blue to yellow water makes it less yellow (to green then to more blue)	**G) Topic:** if we interpret addition as “getting more,” then adding negative is the same as more negative and adding a positive is the same as more positive **Explore:** move your game piece on horizontal number line board more positive (toward “positive park” or more negative toward “negative nettles”)	**H) Topic:** if we interpret subtraction as “getting less,” then subtracting a negative is the same as less negative and subtracting a positive is the same as less positive **Explore:** move your game piece on horizontal number line board less positive (away from “positive park” or less negative away from “negative nettles”)
Integer Operations (n = 21)	**G) Topic:** if we interpret addition as “getting more,” then adding negative is the same as more negative and adding a positive is the same as more positive **Explore:** move your game piece on horizontal number line board more positive (toward “positive park” or more negative toward “negative nettles”)	**H) Topic:** if we interpret subtraction as “getting less,” then subtracting a negative is the same as less negative and subtracting a positive is the same as less positive **Explore:** move your game piece on horizontal number line board less positive (away from “positive park” or less negative away from “negative nettles”)	**Topic:** (review) if we interpret addition as “getting more” and subtraction as “getting less,” then adding a negative is the same as subtracting a positive and vice versa **Explore:** move your game piece on a horizontal number line board more positive/negative or less positive/negative	**Topic:** (review) if we interpret addition as “getting more” and subtraction as “getting less,” then adding a negative is the same as subtracting a positive and vice versa **Explore:** move your game piece on a horizontal number line board more positive/negative or less positive/negative

Data Analysis

Data analysis for pre- and post-test interviews took place at both the task level (focusing on correct/incorrect responses) and composite level (focusing on qualitative differences in responses). Data included students' written answers as well as transcripts of any verbal information they provided when solving the problems in the booklet. Following the task- and composite-level analyses, I analyzed the changes in students' mental models exhibited from the pre- to post-test interviews.

Task Level

Each of the order and value and directed magnitude tasks were scored for correctness. On the counting task, a correct answer involved students counting into the negatives (e.g., "…one, zero, negative one"). Incorrect responses included skipping a number or inserting an additional number. As with counting, when filling out the number lines, students needed to include the negative integers in the correct order. Students could create their own notation for negatives as long as they verbally labeled them as such; in one case, a student wrote "1" but called it "negative one," so I counted it as correct. If students swapped the positive and negative integers (e.g., 2, 1, 0, -1, -2), I counted this as correct. For the ordering task, students had to correctly order numbers from least to greatest or greatest to least, and for the follow-up greatest/least questions, they had to answer *both questions* correctly for each of the two orderings. Similarly, on the greater task, students had to identify all of the greater integers. Finally, each directed magnitude question was scored separately. As long as students indicated the correct direction of movement, the answer was considered correct. Because the answers at the task level were either correct or incorrect, based on the criteria just outlined, a second scorer was not deemed necessary.

Composite Level: Integer Order and Value

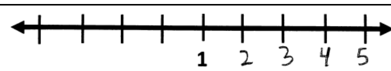
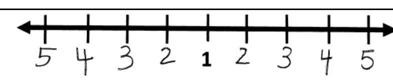
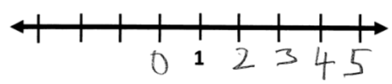
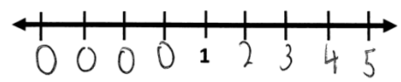
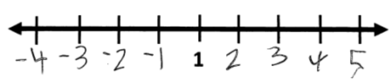
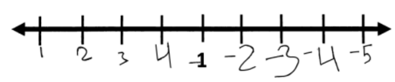
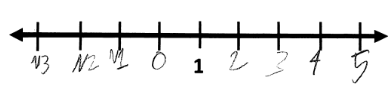
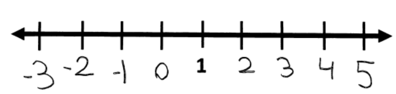
To identify students' integer order and value mental models, I first coded students' responses for each category of the order and value questions based on their qualitative pattern of responses on the items; if students ordered numbers or determined values two different ways, they were coded separately. The codes distinguished between students who treated all numbers as positive, who used negative numbers in non-traditional ways, and who used negative numbers in conventional ways (see Table 5). For example, a student who counted back to *one*, filled in only *whole numbers* on the number line, ordered one set of cards as *whole numbers* and the second set correctly as *integers*, and relied on *absolute value* to determine which number is greater would have the following response pattern classification: one/zero; positive/whole; whole numbers, integer; absolute value.

Within each category it was possible for students to give atypical responses that did not fit their pattern of responses to the other questions. For instance, when filling in the numbers on the number line, a student might have used only positive numbers but written them as continuing on the left side of the number line (e.g., 9, 8, 7, 6, 1, 2, 3, 4, 5). However, the student might not have considered the 6 to be greater than the 8 on the subsequent value questions. In this case, I would have coded the number line as "positive" and made note of the response about the greater value. When counting backward, a few students skipped zero; similarly, some students did not include zero on their number lines. On the ordering task, a few students correctly ordered the cards in pairs with space separating each group (e.g., -9, 3; -5, 8; -7, 0) instead of as one continuous group or switched two positive numbers (e.g., -9, -5, -3, 0, 3, 2, 8). A few others correctly determined which of two integers was greater on all but one problem, and another student called negatives "halves" when counting backward (e.g., "five, four, three, two, one, zero, one and half, two and a half") but then explained that other people call them negatives. I allowed for these single deviations (Vosniadou & Brewer, 1992).

A graduate student double-coded all responses for 12 students (chosen to represent a sample of less straight-forward responses), and we agreed on 93% of the responses coded. The responses on which we differed were due to lack of clarity in the coding descriptors, which I modified. Because all disagreements were surface-level issues, no changes were made in the coding for the other students.

Table 5

Codes and Examples for Order and Value Questions

Codes	Examples	
1. Counting Backward (to..)		
a. One/Zero (Multiple)	"5, 4, 3, 2, 1" "5, 4, 3, 2, 1, 0" or "5, 4, 3, 2, 1, 0, 0, 0"	
b. Zero as Negative	"5, 4, 3, 2, 1, 0, -0"	
c. Reversed Negative	"5, 4, 3, 2, 1, -100"	
d. Negative/Minus	"5, 4, 3, 2, 1, 0, -1, -2…" (may say "minus" or "negative")	
2. Number Line		
a. Positive/Whole Number	 (ends at one)	 (symmetric, missing 0, -1)
	 (ends at zero)	 (multiple zeroes)
b. Negative Alternative	 (symmetric, missing "0")	 (repeated, swapped)
c. Integer (Symbolic)	 (symmetric, own notation)	 (symmetric, traditional notation)
3. Order		
a. Whole Number	Integer cards ordered: 0, 2, -3, 3, -5, 8	
b. Whole-Negative	Integer cards ordered: 0, 2, 3, 8 and -3, -5, -9	
c. Mixed	Negative and positive interspersed: e.g., -1, 6, -6, 4, 3, -7	

d. Zero-Neg-Pos	Integer cards ordered: 0, -3, -5, -9, 2, 3, 8
e. Negative Reversed	Integer cards ordered: -3, -5, -9, 0, 2, 3, 8
f. Integer	Integer cards ordered: -9, -5, -3, 0, 2, 3, 8
4. Greatest/Least; Greater	
a. Positive	In order: 0, 2, -3, 3, -5, 8, -9, the 2 (not 0) is considered least
b. Positive Whole Number	In order: only whole numbers used to determine greatest/least
c. Absolute Value	-9 > 3; -7 > -2 (Judge value based on absolute value)
d. Integer as Zero	3 > -9; -7 and -2 = 0 (Judge negative numbers as worth zero)
e. Integer Reversed	3 > -9; -7 > -2 (Judge negatives based on absolute value)
f. Integer	3 > -9; -2 > -7

Composite Level: Directed Magnitude

To identify students' directed magnitude mental models, I first grouped together similar response patterns according to which questions and how many questions they answered correctly; high/low questions (position of cat on stairs) were analyzed separate from positive/negative questions (starting at zero, move to tick mark on number line more/less positive/negative). Because determining students' response pattern to the directed magnitude questions did not require qualitative judgment, a second scorer was not deemed necessary for grouping these items. The codes for each category provide a description for each response pattern (see Table 6). For example, students who moved the cat up the stairs for all problems were classified as *More, High, Positive*. The *Direction Opposite* response pattern has this name because students moved the cat in the direction (e.g., low, high) opposite of what was given and ignored the comparatives more or less. The *Direction Only* label describes the response pattern where students moved the cat in the given direction but ignored the comparatives more or less.

Table 6

Codes and Examples for Directed Magnitude Questions with Incorrect Answers Italicized

Codes	More High/ More Positive: **Up**	More Low/ More Negative: **Down**	Less High/ Less Positive: **Down**	Less Low/ Less Negative: **Up**

Inconsistent, one correct	Up	*Up*	*Up*	*Down*
	Down	Down	*Up*	*Down*
More, High, Positive	Up	*Up*	*Up*	Up
Less, Low, Negative	*Down*	Down	Down	*Down*
Direction Opposite	*Down*	*Up*	Down	Up
Direction Only	Up	Down	*Up*	*Down*
Inconsistent, three correct	Up	*More*	Down	Up
	Up	Down	*Up*	Up
	Up	Down	Down	*Down*
All correct	Up	Down	Down	Up

Changes in Mental Models

After coding all student responses for the order and value questions, I grouped students according to those with similar patterns of codes starting with their codes to the value questions. Then, I analyzed each resulting pattern of codes to characterize the type of thinking students exhibited based on Vosniadou and Brewer's (1992) mental model categories: initial, synthetic, and formal. However, some coding patterns represented thinking that partially aligned with both initial and synthetic mental models or synthetic and formal mental models, so I created additional "transition" categories, which are reported below. I followed the same process for the directed magnitude responses.

To analyze the impact of instruction on whether students provided responses consistent with having formal mental models, I compared the proportions of students exhibiting formal mental model responses before and after instruction for each group using a McNemar test for change (Dimitrov, 2008) with the Yates correction for continuity[7]. To further analyze the impact of instruction on changes in students' mental models, I organized the students according to their instructional groups and recorded how large their mental model shifts were from pre- to post-test

[7] The McNemar test is used to assess changes in proportions when a dichotomous variable (e.g., passed, did not pass) is involved. The Yates correction for continuity is used if any amounts in the cells are less than 5.

for order and value and directed magnitude. Therefore, a student who was classified as using an *initial* order and value mental model (whole number or absolute value) at pretest and a *transition II* mental model (dual value or unstable integer) at posttest had a shift of 3. Because there were no transition mental models for directed magnitude, a shift of 1 corresponded to transitioning from the use of *initial* to *synthetic* or *synthetic* to *formal* mental models.

To test whether the average differences in mental model advancement were due to the instructional treatments versus chance, I used the Kruskal-Wallis one-way ANOVA by ranks (Shavelson, 1996). Students who reached ceiling (i.e. exhibited formal mental models) on the pretest were eliminated from this analysis. For the directed magnitude mental models, both analyses were conducted for the high/low questions only because the positive/negative questions were not posed on the pretest.

Results

The results are presented in three sections. In the first two sections, I describe the mental models that characterize students' response patterns for integer order and value and directed magnitude, respectively. In the third section, I report the changes in the mental models students exhibited on the pre- and post-tests.

Mental Models of Integer Values and Order

The process of grouping students according to those with similar patterns of codes on the order and value questions resulted in seven groups based on how they interpreted the value of integers primarily and the order of integers secondarily (see Table 7); these groupings were informed by Vosniadou and Brewer's (1992) categories. I interpreted response patterns whereby students consistently relied on whole number principles and treated negative numbers as having positive number values as indicating the use of initial mental models. Response patterns in which students consistently treated negative numbers as having different values than positive numbers but with idiosyncrasies indicated the use of synthetic mental models. Finally, I interpreted students' correct responses across all tasks as indicating the use of formal mental models.

Table 7

Integer Value and Order Mental Models

Order & Value Mental Models		General Description
Initial:	Whole number	Negative integers ignored or treated as positive integers
	Absolute Value	Negative integers ordered but given positive values
Transition I:	Conflicted Value	Negative integers treated as negative *and* positive
Synthetic:	Magnitude	Negative integers reversed in order and value
Transition II:	Dual Value	Negative integers reversed *and* also given correct values
	Unstable Integer	Negative integers used correctly with 2 inconsistencies
Formal:	Integer	Negative integers symmetric around 0 with positive integers

Initial mental models. Students' initial mental models were characterized by two ways of thinking: whole number and absolute value. Students using Initial: Whole Number mental models did not distinguish between positive and negative numbers; they treated all numbers as positive and often ignored the negative signs completely. For example, on the pretest, one student counted backward, "Five, four, three, two, one." He added zero after prompting and said there would not be anything else before that. Consistent with this response, he filled in only positive numbers on the number line. He ordered the first set of integers by absolute value: 0, 2, 3, -3, -5, 8, -9 and explained, "Zero, one, two. Two, three, then four here (points to the space between -3 and -5), five, six, seven, eight." He used the principle that "numbers can be ordered by means of their position in the count list" (Vosniadou et al., 2008, p. 11) to order the numbers. On the number comparisons, he talked about the numbers as if they were positive and chose the greater of two integers based on this information.

Students' using Initial: Absolute Value mental models ordered negative numbers (although sometimes in nonstandard ways, such as ordering them backwards) on at least one task (counting backward, filling in the number line, or ordering integers) but determined which numbers were greatest/least as if the integers had whole number values. For example, on the posttest, a student ordered the integer cards correctly (e.g., -8, -4, -2, 1, 5, 7) but then said that "eight" was the greatest and "one" was the least. When asked what negative numbers are or how they are different than positive numbers, these students typically indicated that negative numbers have negative signs.

One interesting response among students who exhibited the use of initial mental models (whole number or absolute value) involved writing repeated zeroes on the number line or counting backward to never-ending zeroes (e.g., "5, 4, 3, 2, 1, 0, 0, 0, 0"). This was only exhibited by six students, but it does point to the role that students' understanding of zero likely plays in their developing understanding of negative integers.

Transition I mental models: Conflicted value. Students exhibiting the use of conflicted value mental models sometimes correctly determined which integer was greatest/least, especially when comparing a positive and negative integer pair. However, they incorrectly treated negative numbers as equivalent to their positive counterparts or zero, depending on the task. On the pretest, one student counted back to zero and wrote only the digits 0–5 on her number line. She ordered the negative integer cards before 0 but backwards: -3, -5, -9, 0, 2, 3, 8. Then, she said that eight was the greatest and zero was the least because, "Nine minus nine is zero and five minus five: zero, and three minus three: zero." Although she correctly determined that 3 > -9, when comparing -7 and -2, she said that -7 is greater because "Seven is greater than two" and then added, "But they're both zero." Students exhibiting this model of thinking did not rely solely on their whole number principles nor did they consistently treat negative integers as different from positive integers. I characterize this type of thinking as being in transition toward using a synthetic mental model of integers.

Synthetic mental model: Magnitude. Students exhibiting the use of this mental model consistently demonstrated an understanding that negative numbers are less than zero but treated negative integers with larger absolute values as greater than those with smaller absolute values. When ordering the integer cards, they put the negative integers in reverse order, and when choosing which integer is greater, they correctly identified that 5 > -7, 4 > -8, and 5 > -9; however, they incorrectly thought that -3 > -1, -6 > -2, and -8 > -6. When describing why -6 is greater than -2, one student explained, "They're both in the negative part, but this one [-6] is a bit higher in the negative part."

Transition II mental models. Transition II mental models were characterized by two ways of thinking: dual value and unstable integer. Students exhibiting the use of a Transition II: Dual Value mental model appeared to understand that negative numbers are less than zero. They counted into the negatives and correctly included negatives on the number line. They also considered negative integers with the larger absolute value to be greater than those with smaller absolute value about half of the time but correctly determined which was greatest/least the other half of the time, depending on the task. For example, two students correctly determined which of two integers was greater but reversed the negative numbers in the ordering task and did not correctly identify the greatest and least integers in the sets. Others correctly ordered one of the two sets of integers or ordered both sets correctly but then did not identify which integer was greater when they were presented in pairs. Because these students inconsistently identified the correct value of the integers, I considered them to be transitioning to the use of formal integer mental models.

Use of the Transition II: Unstable Integer mental model was evident in only two students. These students correctly determined which integers were greater on the greater task but had more than one acceptable deviation (atypical response) in their answers to the various questions. One student counted into the negatives but then did not mark negatives on her number line. She also ordered the negative integer cards backwards and after the positive numbers (i.e., 1, 5, 7, -2, -4, -8) but was still able to determine which was least and greatest. The other student correctly answered all of the integer order and value questions except she skipped 0 when counting into the negatives *and* left out 0 on her number line (instead she wrote -1 next to 1). Overall, these students were not consistent enough to meet the strict criteria to be characterized as using formal mental models; therefore, they were also considered to be in transition.

Formal mental models: Integer. Students exhibiting the use of formal mental models answered all value and order questions correctly. When explaining which negative integer was greater, they often referred to the opposite relationship between positive and negative numbers. For example, on the posttest, one student said -2 was greater than -6, “because I know negatives, the lesser number (referring to “2” in the -2) is the biggest and the more number goes downer and downer and downer.” Students using a formal mental model also determined which negative integer was greater by thinking about which one was closer to zero as found by Stephan and Akyuz (2012).

Mental Models of Directed Magnitude

I analyzed the seven response patterns identified for the directed magnitude questions (Table 6) in light of Vosniadou and Brewer’s (1992) mental model categories and found they were adequately characterized as exhibiting either initial, synthetic, or formal mental models. I considered students with scores of four out of four to be using a formal mental model. For the other response patterns, I determined whether students’ responses reflected understanding of either magnitude *or* direction (initial mental model) or a combination (synthetic mental model) (see Table 8). Students with three out of four correct had response patterns consistent with switching between a focus on the magnitudes *more* and *less* and the directions *high* and *low* or *positive* and *negative*, suggesting there was some attempt to make sense of both simultaneously. I considered these response patterns (inconsistent, three correct from Table 6) to exhibit the use of a synthetic mental model. All students with at most two out of four correct either had responses patterns that were inconsistent or one-directional (i.e., Inconsistent, one correct; More, High, Positive; and Less, Low, Negative categories from Table 6), or direction-biased, where they consistently interpreted directions *high* and *low* or *positive* and *negative* but not in combination with magnitudes *more* and *less* (i.e., Direction Opposite and Direction Only categories from Table 6). I categorized these responses as exhibiting the use of an initial mental model.

Table 8

Directed Magnitude Mental Models

Directed Magnitude Mental Model		**General Description**
Initial:	One-directional	0, 1, or 2 out of 4 correct: Guesses; only moves in one direction
	Direction-biased	0, 1, or 2 out of 4 correct: Only focuses on the directions involved
Synthetic:	Magnitude-aware	3 out of 4 correct: Interprets *more* in two directions and *less* in one direction OR *less* in two directions and *more* in one direction
Formal:	Directed Magnitude	4 out of 4 correct: Interprets more or less of opposite dimensions

Initial mental models. Students' initial mental models were characterized by two ways of thinking: one-directional and direction-biased. The use of Initial: One-directional mental models were especially evident on the pretest; several students thought the cat on the stairs would go higher for all of the directed magnitudes or lower for all of the directed magnitudes, and others were inconsistent in their responses. Students exhibiting the use of Initial: Direction-biased mental models considered the direction only; they correctly interpreted movements *more high* and *more low* but incorrectly moved the cat higher when told to move it *less* high and moved the cat lower when told to move it *less* low. In one student's case, he interpreted the locations as fixed, "This is high (points to the top of the staircase), this is less high (points just above the middle point), this is less low (points below the middle point), and this [is] low (points to the bottom)." Therefore, when asked to move the cat, which was sitting just above the middle point of the staircase, *less high*, he moved the cat up the stairs; otherwise it would have ended up in his "less low" section.

Synthetic mental model: Magnitude-aware. Students exhibiting the use of synthetic mental models correctly interpreted movements *more* or *less* in one direction and either *more* or *less* in the opposite direction. Most students in this category correctly interpreted *more high* and *less high* as well as *more low*. *Less low* was difficult to interpret because students could not correctly answer it by focusing on either *less* or *low*. Students could focus on either the direction or magnitude and correctly answer the other three categories; however, they had to switch between focusing on either the direction or magnitude. One student preferred to focus on the direction. When asked if a cat sitting just below the middle of the staircase was *less high* or *less low*, he said, "Less low" because "this (points to the bottom half) is the low and this (points to the top half) is the high." Although he correctly moved the cat *less high*, when asked to move the cat *less low*, he dropped the reference to "less" and moved it lower.

Formal mental model: Directed magnitude. I considered students to be exhibiting formal mental models of direct magnitudes when they successfully interpreted movements more or less within either direction. These students understood the opposite nature of the terms *less high* and *less low*. One student explained, "Low is less high…going less low, then you're going up.” Another student expressed the opposite nature in relation to double negatives: “Less low is *not not*. It will be more high.”

Changes in Mental Models

Integer order and value mental models. Table 9 shows the percent of students across the groups exhibiting each of the order and value mental models on the pretest and posttest. None of the students in the Unary Instruction group exhibited the use of initial (whole number or absolute value) mental models on the posttest even though 75% did so on the pretest. In contrast, 20% of the students in the Combined Integer Instruction group, who also learned about integer order and values, and 62% of the students in the Binary Instruction group exhibited initial mental models after instruction. Overall, both Unary and Combined Integer Instruction groups showed significant changes from pre- to post-test.

Table 9

Percentage of Students in Each Instructional Group Using Different Mental Models on Pre- and Post-Tests.

	Pretest			**Posttest**		
Order & Value Mental Models	Unary (n = 20)	Binary (n = 21)	Combined (n = 20)	Unary (n = 20)	Binary (n = 21)	Combined (n = 20)
Formal:						
Integer	10%	14%	10%	**55%**[a]	24%	**45%**[a]
Transition II:						
Unstable Integer	0%	0%	0%	10%	0%	0%
Dual Value	5%	0%	0%	10%	0%	10%
Synthetic:						
Magnitude	0%	5%	0%	5%	14%	0%
Transition I:						
Conflicted Value	10%	0%	15%	20%	0%	25%
Initial:						
Absolute Value	5%	10%	5%	0%	14%	15%
Whole Number	**70%**	**72%**	**70%**	0%	**48%**	5%

[a]Indicates which groups had a significant number of students exhibit the use of formal mental models according to the McNemar test, see Appendix B.

Table 10 illustrates in more detail how students' mental model classifications changed. None of the students moved backward in their mental model classification. However, six students (30%) in the Combined Integer Instruction group, two students (10%) in the Unary Instruction group, and seventeen students (81%) in the Binary Instruction group exhibited the use of the same mental model on the pretest as they did on the posttest. In the case of the Unary Instruction group, these two students exhibited the use of formal mental models on both tests. Similarly, three of the seventeen students in the Binary Instruction group and two of the six students in the Combined Integer Instruction group also experienced this ceiling effect.

Table 10

Groups' Shifts in Order and Value Mental Model Categories Between Tests.

Mental Model Shifts for Instructional Groups (Pre- to Post-test)	**Shift of 0** I-I, T_1-T_1, S-S, T_2-T_2, F-F	**Shift of 1** I-T_1, T_1-S, S-T_2, T_2-F	**Shift of 2** I-S, T_1- T_2, S-F	**Shift of 3** I-T_2, T_1-F	**Shift of 4** I-F
Unary Instruction (n = 20)	10%	25%	10%	20%	35%
Binary Instruction (n = 21)	81%	0%	10%	0%	10%
Combined Instruction (n = 20)	30%	30%	5%	10%	25%

Note. Each cell shows the percentage of students in each instructional group who shifted classification categories in their mental models from pre- and post-test, where I = Initial, T_1 = Transition I, S = Synthetic, T_2 = Transition II, and F = Formal. A student who falls in the "Shift of 0" category demonstrated the same mental model on both tests.

Results of the Kruskal-Wallis ANOVA indicate a significant effect for instructional group ($H_{observed}$ = 15.9, $H_{critical}$ = 5.99, $\alpha < .05$). Pairwise comparisons reveal that the Unary Instruction (mean rank = 37) and Combined Integer Instruction (mean rank = 30) groups improved significantly in terms advancing toward using formal integer order and value mental models compared to the Binary Instruction group (mean rank = 16) (HSD = 12.27, $\alpha < .05$). There was no significant difference between the Combined Integer Instruction and Unary Instruction groups.

Directed Magnitude Mental Models. Table 11 shows the percent of students across the groups exhibiting the use of each mental model for directed magnitude on the pre- and post-tests. The majority of students were magnitude-aware on the pretest. However, only the Binary and Combined Integer Instruction groups showed significant growth by exhibiting the use of formal mental models on the posttest.

Table 11

Percentage of Students in Each Instructional Group With Different Directed Magnitude Mental Models (for More and Less High and Low) on Pre- and Post-Tests

Directed Magnitude Mental Model (High and Low)	**Pretest**			**Posttest**		
	Unary (n = 20)	Binary (n = 21)	Combined (n = 20)	Unary (n = 20)	Binary (n = 21)	Combined (n = 20)
Formal: Directed Magnitude	10%	5%	5%	15%	**38%**[a]	**45%**[a]
Synthetic: Magnitude-Aware	**45%**	**52%**	**65%**	**65%**	29%	35%
Initial: Direction-Biased	25%	29%	0%	10%	29%	0%
One-Directional	20%	14%	30%	10%	5%	20%

[a]Indicates which groups had a significant number of students exhibit use of formal mental models according to the McNemar test, see Appendix.

All instructional groups had similar total shifts in their mental models (see Table 12), so results of the Kruskal-Wallis ANOVA indicated no significant effect for instructional group ($H_{observed} = .54$, $H_{critical} = 5.99$, $\alpha < .05$). However, as seen from Table 12, more of these shifts occurred to formal mental models for the Binary and Combined Integer Instruction Groups.

Aside from not shifting, most groups had a shift of one category from pre- to post-test. Five of the six students in the Combined Integer Instruction group and six of the seven students in the Binary Instruction group who provided responses characteristic of having synthetic mental model on the pretest were classified under formal mental model on the posttest. However, six of the seven shifts of 1 for the Unary Instruction group were from initial mental models to synthetic mental models.

Table 12

Groups' Shifts in Directed Magnitude Mental Model Categories Between Tests

Mental Model Shifts from Pre- to Post-test	**Shift of -2** F-I	**Shift of -1** F-S, S-I	**Shift of 0** I-I, S-S, F-F	**Shift of 1** I-S, S-F	**Shift of 2** I-F
Unary Instruction (n = 20)	0%	5%	60%	35%	0%
Binary Instruction (n = 21)	5%	0%	52%	33%	10%
Combined Instruction (n = 20)	0%	10%	45%	30%	15%

Note. Each cell shows the percentage of students in each instructional group who shifted classification categories in their mental models from pre- and post-test, where I = Initial, S = Synthetic, and F = Formal. A student who falls in the "Shift of 0" category demonstrated the same mental model on both tests.

Although students did not answer the directed magnitude questions related to positive and negative direction on the pretest, the posttest results indicate that these directed magnitude questions were easier for students to interpret than those for high and low (see Table 13). Further, the majority of students in the Binary and Combined Integer Instruction groups, who played games where they moved more and less positive and negative on number lines, had response patterns consistent with a formal mental model for directed magnitudes. The Unary Instructional group had fewer students who correctly interpreted "less negative" (55%) as compared to the Binary Instruction group (76%) and the Combined Integer Instruction group

(70%), which appears to be the main reason why they did not have as many students exhibiting responses consistent with a formal mental model of directed magnitude on these items.

Table 13

Percentage of Students in Each Instructional Group with Different Directed Magnitude Mental Models (for More and Less Positive and Negative) on the Posttest

Directed Magnitude with Positive and Negative	Unary Instruction n = 20	Binary Instruction n = 21	Combined Instruction n = 20
Formal: Directed Magnitude	45%	**71%**	**60%**
Synthetic: Magnitude-Aware	15%	5%	15%
Initial:			
Direction-Biased	35%	24%	25%
One-Directional	5%	0%	0%

Discussion

The results of this study detail first graders' mental models of negative integers and show how instruction on the binary and/or unary meanings of the minus sign supports their thinking about negative integers.

Students' Mental Models

Students in this study exhibited a range of integer understanding that highlights important conceptual changes the first graders had to overcome before developing formal mental models.

Initial mental models. As suggested by Vosniadou and Brewer's (1992) theory, some students' interpretations of integers are heavily constrained by their framework theory of whole numbers. In this study, students classified as exhibiting initial integer mental models interpreted numbers and the meaning of the minus sign from a positive perspective. Their whole number conceptions of zero as the smallest number or "nothing" appeared to constrain their ability to accept other numbers as less than zero and may clarify why some students order negative integers next to their positive counterparts, as seen by Peled et al. (1989). Students reasoning from a positive perspective might either ignore the negative sign or rely on the binary interpretation of the minus sign and interpret negative integers as numbers that *will be* taken away. Likewise, they might interpret all directed magnitudes as referring to the same (usually positive) category, such as “more,” “high,” or “positive,” similar to the finding of Whitacre et al. (2012b) that students often interpret integer word problems from a positive perspective.

Some students classified as exhibiting absolute value mental models appeared to attribute new values to the negatives in that they correctly named or ordered a set of given integers. However, the results from this study extend the findings of other researchers (e.g., Aze, 1989; Bishop et al., 2011; Wilcox, 2008) by demonstrating that students do not necessarily attribute the “new numbers” with values less than zero; rather, students’ actions may be more consistent with sorting the integers based on surface features—as if distinguishing between a red five and a green five.

Other students interpreted the new notation more consistently by considering negative integers as numbers that *have been* taken away, supporting previous findings (Bofferding, 2010; Hughes, 1986). They interpreted the binary meaning of the minus sign as indicating a subtraction action that has already been completed and treated negative integers as equivalent to zero in both value and order, which is similar to the findings of Schwarz et al. (1993). However, the results presented here also suggest that some students consider zeroes to continue indefinitely. A few students even correctly interpreted the values of negative numbers but left out zero when counting backward and labeling their number lines. It is possible that they considered negatives to be more specific forms of zero. In any case, these responses indicate the strength of students' zero conceptions and their reluctance to accept the existence of values less than zero.

Synthetic mental models. Although previous research indicates that students either have difficulty making sense of values less than zero or correctly comparing two negative integers (e.g., Hativa & Cohen, 1985; Peled, Mukhopadhyay, & Resnick, 1989), the results of this study provide a more nuanced view and highlight the importance of students' numerical reasoning in relation to zero and the strength of their framework theory for whole numbers. Students' value conceptions appeared to be harder to change than their order conceptions, and their use of categorization language became more pronounced as they grappled with the difference between positive and negative numbers.

Based on their framework theory for whole numbers, some students considered numbers further from zero in the counting sequence or on a number line as "higher" or having greater values. Similar to the findings of Schwartz et al. (1993), although students ordered negative numbers correctly, they considered negatives further from zero or "higher in the negatives" as greater than those closer to zero (e.g., -5 > -2) but less than positives. Students who began to consider *direction* in addition to *magnitude* divided up the number line, much like the Divided Number line model described by Peled et al. (1989); however, these students included more categories, such as "less low" or "less negative," indicating locations close to the middle of the number line or zero but on the "low" or "negative" side. Other students considered numbers higher in the counting sequence or to the right on the number line as having greater values. Students who acknowledged the unary meaning of the minus sign ordered negative numbers in reverse, so that negative integers with higher absolute values were further to the right on the number line (e.g., -8, -9, 0, 1). Both instances reveal that students find ways to coordinate their current understanding of integer order and values.

Formal mental models. As students in this study transitioned from relying solely on their framework theory for whole numbers, their interpretations of the integers sometimes changed depending on whether they were reasoning primarily about integer order, values, or directed magnitudes. Some students correctly reasoned about integer values when presented in isolation while others did better when they could see the numbers laid out in order. However, if students incorrectly ordered the integers, the incorrect orderings often influenced students' integer value judgments.

Students with formal mental models were able to interpret the integers and/or directed magnitudes on a continuum or continuous number line (Peled et al., 1989) rather than only categorically with zero as a boundary point. Those classified as exhibiting formal mental models for integer values and order were able to restructure their framework theory so that zero was no longer considered the smallest number and were able to order the negative numbers as less than zero in conjunction with their expanded count list. On the other hand, students who were classified as exhibiting formal mental models for directed magnitudes restructured their framework theory so that addition makes "bigger" in the positive *or* negative numbers (where bigger means further from zero) and corresponds to movements further toward the positive numbers (e.g., -3 + 1) or negative numbers (e.g., 3 + -1). They also broadened their concept of subtraction to mean getting smaller in the positive *or* negative numbers (where smaller means closer to zero) and corresponding to movements away from the positive numbers (e.g., 3 – 5) or negative numbers (e.g., -3 - - 5).

Instructional Influences on Mental Models

Unary instruction. Students who participated in the unary instruction had the opportunity to confront their understanding of the subtraction sign and reason about its differences from the negative sign. The majority of first graders in this group were able to distinguish between positive and negative numbers and apply this knowledge on subsequent questions, indicating that an instructional focus on the meanings of the minus sign can help students overcome their reliance on the binary meaning of the minus sign as found in Bofferding (2010). Further, unlike students in Peled et al.'s (1989) study, almost all of these first graders could order integers correctly and determine which was greater – especially after instruction. However, reasoning about numbers greater and less than zero in their activities may have also helped solidify zero as a point that divides the number line (Peled et al., 1989). Few students in the unary group reasoned about directed magnitudes on a continuous scale where less positive corresponds to more negative; rather, they interpreted directed magnitudes as locations above or below a middle point.

Binary instruction. After participating in instruction focused on the binary meaning of the minus sign, a large majority of students could distinguish between movements more positive, less positive, more negative, and less negative; they were able to interpret the movements along a continuum across a continuous number line where the subtraction sign indicated a movement less in a direction and the negative sign indicated the referent direction. These results build on Thompson and Dreyfus's (1988) work by demonstrating that younger students can also reason about directed movements involving addition *and* subtraction (moving more *or* less in a particular direction). Having experiences moving through the zero point may have helped them overcome the need to form specific locations for these directed magnitudes. However, the results were strongest for the directed magnitudes that students practiced; they did not have experiences moving “less low” during the instructional activities, and only about a third of students could correctly interpret it. Further, the meaning of the subtraction sign was kept consistent with students’ ideas of getting less of something, in contrast to traditional integer instruction in which subtraction means “going backward” (Liebeck, 1990) or “facing left” (Moreno & Mayer, 1999), and the minus sign is interpreted as opposite. Unfortunately, the instruction did not help most of these students understand the order or values of the negative integers even though they could identify the negative direction and read the integer names on their game cards.

Combined integer instruction. Students in the combined integer instruction group experienced the benefits of both the unary and binary instruction. Overall, the results of the instruction indicate an improvement on traditional number line-related instruction because the combined instruction supported students’ developing understanding of integer order, values, and directed magnitudes *with* a particular focus on the various meanings of the minus sign that is lacking in most integer instruction.

Connections to the Theoretical Framework

According to the Case’s (1996) model, a number of elements are coordinated when students develop a CCSN: order, value, addition and subtraction understanding, and notation. In the same way that Case (1996) suggested that students use the value and order of positive numbers separately before coordinating this knowledge into the CCSN, students in this study who exhibited the use of initial mental models (specifically the absolute value model) ordered negative numbers correctly (although sometimes separate from positives) but did not use this information to reason that the values would also be different. Others ordered the numbers incorrectly but still selected the correct least and greatest numbers. Such responses suggest that as students learn about integers they shift between thinking about integer values *or* order (and the principles underlying these ideas) before they consider both simultaneously.

Further, students who exhibited thinking consistent with formal mental models of integer value and order also did not always do so for directed magnitude (and vice versa). This is important because most integer instruction does not adequately address ideas around operations in terms of directed magnitudes, so although students may appear to grasp integer concepts because they can answer questions about integer values and order, they may not understand how to interpret moving more or less distance in a positive or negative direction. Such understanding, if developed, could support their understanding of integer addition and subtraction.

Students who interpreted integers in ways consistent with having a formal integer mental model reasoned about the same elements considered important in the CCSN theory, reflecting the use of a single, mental number line. These results suggest that Case's (1996) CCSN theory can accommodate the domain of integers at the uni-dimensional level. If this is the case, then it would indicate that students can make significant domain-related changes within Case's levels and would provide insight into what types of changes we might expect students to be able to make and when.

Limitations and Future Research

One limitation of this study is that students' understanding of directed magnitudes with positive and negative directions were not tested on the pretest. Although it is unlikely there were significant differences between the groups initially, a pretest measure would have helped clarify how much students' ideas about moving more and less positive and negative developed. Moreover, to increase the reliability and validity of the construct, further investigation into students' directed magnitude mental models should include more diverse test items to capture students' understanding of directed magnitudes in terms of operations and comparing object positions.

Further, future research and analyses should clarify the relationships among students' mental models and their arithmetic solutions, especially regarding how they interpret the meanings of the minus sign. In some countries, such as Japan, students are taught to call the negative sign "minus" (A. Murata, personal communication, December 23, 2011). Use of this language may reflect the student's family context and background, but the impact of using "negative" versus "minus" language needs to be explored further in terms of how students interpret integer values and directed magnitudes.

This study focused on the use of one instructional model, the number line model; however exploring students' mental models after instruction involving the cancellation model may provide an additional validity check for the mental models described here. Also, given the setup of the study, there was little freedom to follow students' ideas or invented notation, which could lead to other fruitful insights into students' thinking. Future work in this area should capitalize on additional research methods, such as microgenetic investigations (Siegler, 1996). By assessing students' thinking frequently as they engage in repeated experiences that are hypothesized to promote cognitive change, researchers can begin to clarify what prompts changes in students' mental models and provide details on whether all students develop synthetic mental models of integers before exhibiting formal mental models or whether only certain experiences prompt these changes.

Implications for Instruction

The integer instruction in this study played an important role in promoting conceptual change, and the data provided can help teachers and researchers understand what concepts students need more experiences with if they exhibit some of the responses common to the different mental models discussed here. Past research has emphasized the difficulty students have making sense of the multiple meanings of the minus sign (Bofferding, 2010; Vlassis, 2004, 2008); the results here support and add to these findings but with an encouraging twist. The first graders used their binary understanding of the minus sign in useful ways to help them distinguish between positive and negative (or subtracted) numbers. Students who over-rely on the subtraction meaning of the minus sign might benefit from exploring a series of subtraction problems using empty number lines. For example, they could first solve 2 – 1 = 1, 2 – 2 = 0, then 2 – 3 = -1, only using their own notation. Then, they could explore the negative sign in contrast to the binary minus sign, as students did in the Unary and Combined Integer Instruction groups.

Just seeing the order of integers, as did students in the Binary Instruction group, only helped a few of them exhibit conceptual change in terms of their integer order and value understanding and interpretations of the unary meaning of the minus sign. Therefore, in terms of current integer instruction, these results highlight the importance of focusing on the differences between positive and negative integers. The school where the participants came from used the California edition of enVision Mathematics (Pearson Education, 2009) curriculum, and it includes integer instruction in the fourth grade—for most U.S. schools, this is now in the sixth grade. However, there are only three lessons devoted to this topic, with none focused on operating with directed magnitudes or making sense of the multiple meanings of the minus sign. Helping students interpret the integers along a continuum is important for building up the conception that subtracting a negative number (or getting less negative) is equivalent to adding a positive number (or getting more positive). Without these experiences, it is likely that many students would continue to struggle with the concepts. Given the success of the instructional interventions discussed here, it would be worthwhile to explore the use of similar instruction with older students, providing them with more targeted integer experiences around the multiple meanings of the minus sign and interpreting integer values from both positive and negative perspectives.

Supporting Conceptual Thinking

Although these suggestions could help students in the upper-elementary or middle grades, there are conceptual reasons why delaying the introduction of negatives numbers is potentially problematic. As expressed by Ashvin in the opening quote of the article, a late introduction to negative numbers can leave students feeling frustrated with having to relearn addition and subtraction with a new set of numbers. Even if older students learn the difference between the unary and binary meanings of the minus sign, they may resist applying the knowledge to situations long-engrained in their practice, such as always subtracting with the larger number first (Murray, 1985). Some people will resort to memorizing and using rules for solving integer problems, despite not knowing why they work (Bofferding & Richardson, 2013). Further, students who continue to associate *more* with moving right on the number line (as in whole number addition) and *less* with moving left on the number line (as in whole number subtraction) will have difficulty making sense of operations with negative numbers, just as many students in the Unary Group had difficulty interpreting the directions of *less low or less negative.*

The results of this study indicate that first graders can reason about getting more positive, less positive, more negative, and less negative, which builds on the language they use for interpreting whole number addition and subtraction (Vosniadou et al., 2008) and could support their reasoning about integer operations. However, students need more time than currently provided in instruction to work with negative integer concepts. As students develop an understanding of negative integers, they need more opportunities to confront their changing understanding of the minus sign. An earlier introduction to negative integers could provide deeper meaning and coherence to the mathematics that students explore throughout the elementary grades (e.g., the meaning of addition, the role of zero). Therefore curriculum developers and policy makers may want to reexamine the order and relevance of integer topics more carefully. The findings of this study have shown that students are quite capable of learning about integers much earlier than fifth grade.

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Appendix A—Interview Protocol

Words in italics are said by the interviewer.

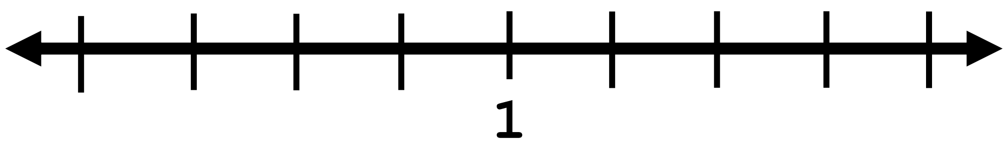
Warm Up: Operations

What is addition? **If child is unsure:** *What does it mean to add? (What happens when you add?)*
What is subtraction? **If child is unsure:** *What does it mean to subtract? (What happens when you subtract?)*

Counting Backward

Start at 5 and count backwards as far as you can.
If child stops at 1: *Can you count backwards any further? Is there anything less than 1?*
If child stops at 0: *Can you count backwards any further? Is there anything less than zero?*
(Student answers) *How do you know?*
If child stops at a negative number: *Can you go any further?*
If child indicates the numbers keep going: *What would the last number be?*

Filling in a Number Line



(Point to number line.) *Fill in the rest of the numbers on the number line shown here.*
If child only labels one side of the 1: *Does anything go over here (point to other side)?*
If child doesn't understand what to do: *If I were counting up, what would come after 1?*
If I were counting down, what would come before 1?

Ordering Integers & Greatest/Least

(Lay down the integer cards in front of child in the given order.) *Put the following number cards in order from least to greatest.*
After child finishes: *How did you know how to put them in order? Which is greatest? How do you know? Which is least? How do you know?*

<u>Pretest Items</u>		<u>Posttest Items</u>	
a) 2, -3, 0, -9, 3, 8, -5	b) 6, -6, 4, -7, 3, -1	a) 5, -2, -8, 1, 7, -4	b) 3, 8, -9, -7, 0, -5

Greater Integer

(Show the two numbers to compare.) What are these numbers? Circle the greater number. How do you know it's greater?

If student gets 1st one wrong, give the two extra positive-positive problems (designated by *)

Pretest Items		Posttest Items	
1) Show 8 and 6.	* Show 1 and 7.	1) Show 6 and 4.	* Show 3 and 5.
2) Show 3 and -9.	* Show 4 and 2.	2) Show 5 and -7.	* Show 4 and 2.
3) Show -2 and -7.		3) Show -3 and -1.	
4) Show -5 and 3.		4) Show -8 and 4.	
5) Show -8 and -2.		5) Show -6 and -2.	

Greater Integer (Story Problem: Who is Winning?)

(Show the two numbers to compare.) Two children are playing a game and trying to get the highest score. Who is winning? How do you know?

If student doesn't mention how many more points winner is ahead:
How many points does <name of losing person according to student> need to get to have the same number of points as <name of winning person according to student>?

Pretest Items		Posttest Items	
1) Abigail: 4	2) Crystal: -7	1) Amy: 5	2) Dan: -8
Joseph: -7	Leon: -3	Ken: -9	Will: -6

Numbers and Signs

First two problems: *Circle all of the numbers in the equation. Read the numbers you circled. How did you decide what to circle?*

Second two problems: *Circle all of the plus signs and minus signs which tell you whether to add or subtract. How did you decide what to circle?*

Pretest Items		Posttest Items	
Numbers	*Plus and Minus Signs*	*Numbers*	*Plus and Minus Signs*
Show -5 + 3 − -3 = 1	Show -4 − 3 + -1 = -8	Show -2 + 4 − -5 = 7	Show -5 − 3 + -1 = -10
Show 6 + -2 − 7 = -3	Show 5 − -3 + 2 = 10	Show 3 + -6 − 8 = -11	Show 3 − -2 + 4 = 6 + 3

Commutative?

Look at the problems. You don't have to solve them, but will these problems give you the same answer? How do you know?

Pretest Items		Posttest Items	
4 + 5 5 + 4	3 − 1 1 − 3	2 + 5 5 + 2	4 − 1 1 − 4
6 − 4 7 − 4	5 − 8 8 − 5	7 − 3 7 − 4	2 − 9 9 − 2
3 + 2 3 + 3	9 − 6 6 − 9	6 + 3 2 + 6	0 − 8 8 − 0

Bi-dimensional Subtraction

Go ahead and solve this problem. How did you solve it?

Pretest Items		Posttest Items	
a) 23 − 15	b) 31 − 27	a) 24 − 16	b) 23 − 17

Operations with Directed Magnitude

(Show picture of cat on stairs) *Put an X on the stair where the cat will be if she moves [insert number here] stairs [insert direction here]? How did you solve it? Is the cat [more/less high or more/less low]?*

Pretest Items		Posttest Items	
2, less high	3, less low	1, more low	2, less high
1, more low	7, more high	4, more high	3, less low

Integer Problems (random order)

(Show problem.) *Go ahead and solve this problem. How did you solve the problem?*

If child says, "I don't know" or says, "I want to skip it.": *Do you want to take a guess?*

If child guesses: *How did you decide what to guess?*

(Otherwise move on to the next problem.)

5 - -3	4 - -5	-4 - -7	-8 - -8	-7 + -1	-3 + 1	3 - 9	-5 - 9
8 - -7	6 - -7	-2 - -6	-5 - -5	-6 + -4	-4 + 6	6 - 8	-3 - 5
9 - -2		-6 - -9	-4 - -3	7 + -3	-1 + 8		1 - 4
			-8 - -5	5 + -2	-9 + 2		
					-2 + 7		

Posttest Positive/Negative Directed Magnitude Items

	(Show a number line with only 0 labeled.) 1) *Put your finger at zero. Move your finger more positive 2 lines and draw a triangle.* 2) *Put your finger at zero. Move your finger more negative 4 lines and draw a square.* 3) *Put your finger at zero. Move your finger less positive 4 lines and draw a triangle.* 4) *Put your finger at zero. Move your finger less negative 2 lines and draw a circle.*
	Posttest Wrap-up *What are negative numbers?* *What's the difference between a negative number and a positive number?*

Appendix B—McNemar Tests for Change

Table A1

Changes in Formal/Not Formal Integer Order and Value Mental Models from Pre- to Post-test

Does the student have a formal mental model of integer order and value?

Combined Instruction, N = 20

	Posttest NO	Posttest YES
Pretest NO	11 (55%)	**7 (35%)**
Pretest YES	0 (0%)	2 (10%)

$\chi^2(1, N = 20) = 6.04, p < .05$

Unary Instruction, N = 20

	Posttest NO	Posttest YES
Pretest NO	9 (45%)	**9 (45%)**
Pretest YES	0 (0%)	2 (10%)

$\chi 2 \ (1, N = 20) = 8.03, p < .01$

Binary Instruction, N = 21

	Posttest NO	Posttest YES
Pretest NO	16 (76%)	2 (10%)
Pretest YES	0 (0%)	3 (14%)

$\chi^2(1, N = 21) = 1.13$, not sig.

Table A2

Changes in Formal/Not Formal Directed Magnitude (low/high) Mental Models from Pre- to Post-test

Does the student have a formal mental model of directed magnitude with high and low?

Combined Instruction, N = 20

	Posttest NO	Posttest YES
Pretest NO	11 (55%)	**8 (40%)**
Pretest YES	0 (0%)	1 (5%)

$\chi^2(1, N = 20) = 7.03, p < .01$

Unary Instruction, N = 20

	Posttest NO	Posttest YES
Pretest NO	17 (85%)	1 (5%)
Pretest YES	0 (0%)	2 (10%)

$\chi^2(1, N = 20) = 0.25$, not sig.

Binary Instruction, N = 21

	Posttest NO	Posttest YES
Pretest NO	12 (57%)	**8 (38%)**
Pretest YES	1 (5%)	0 (0%)

$\chi^2(1, N = 21) = 4.69, p < .05$